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ON RADICALS IN MOBS

by

© KAR PING SHUM M. SC. (LEEDS)

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Abstract

This dissertation is a study of radicals of ideals in mobs. The study of mobs began about 1950, and Professor A.D. Wallace is known as the founder. He has originated and contributed heavily to most of the major areas of research in this field, especially the ideal theory. Our purpose is, in some sense, to give generalizations of the results of A.D. Wallace and others, by considering radicals of ideals instead of ideals alone. By a mob, we shall mean a Hausdorff semigroup.

In Chapter I, we introduce the concept of algebraic radicals in abelian mobs and study the stability of such radicals. We prove that under some special conditions, any open ideal A of a mob S is radically stable without requiring the algebraic radical of A to be closed. Relations of a compact group and the boundary of the algebraic radical of A are also investigated. Theorems concerning the reducibility of ideals in mobs are obtained.

Topological radicals in compact abelian mobs are treated in Chapter II. We prove that if the topological radical is dense in a compact mob S, then S has no local zeros, and if S contains zero and local zeros, then S must be disconnected.

Some conditions which leads to the existence of a local zero in a compact mob are given. A characterization of compact abelian mobs is obtained.

In Chapter III, we extend our studies to radicals of non-abelian mobs. The Wedderburn radical, e-invariant radical and Thierrin radical in mobs are studied. Some results of their radicals in ring theory are transferred to mobs. We show that in a divisible mob without zero, its e-invariant radical can be a compact connected group.

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Background knowledge

A topological semigroup is an ordered triple $(S,\mathcal{J},m)$ such that (i) $(S,\mathcal{J})$ is a non-empty Hausdorff space

(ii) m is a continuous function from $S \times S$ into S such that $m(x,m(y,z)) = m(m(x,y),z)$ for all x,y and z in S. Following accepted custom, we shall shorten $(S,\mathcal{J},m)$ to S, $m(x,y)$ to $x*y$ and topological semigroup to mob. The following shown in Figure I are some simple examples of mobs, and with their usual multiplication: the interval $[0,1]$, the unit disk D in the complex plane, and, for fixed $n > 1$, any convex subset of D which contains the n^{th} roots of unity $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. (We sketch an illustration of this for $n = 3$)

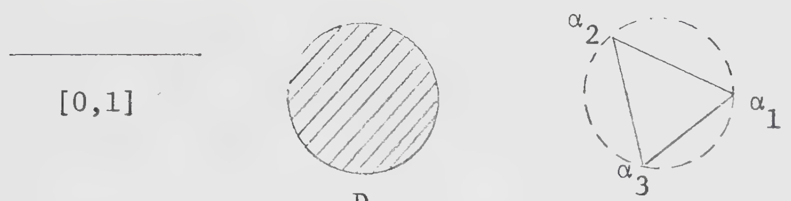


FIG. I

Let S be a mob. An element e in S is called an idempotent if $e.e=e$. We let $E(S)$ denote the set of idempotents of S, and where there is no ambiguity we write E instead of $E(S)$. We remark that E is always a closed subset of S. An idempotent e of S is called Primitive if $f^2 = f \in eSe$ implies $f = 0$ or $f = e$.

An element $t \in S$ is called the zero of S if $xt = tx = t$ for all $x \in S$. It is easily seen that the zero of S if it exists, is unique and is an idempotent. For $A,B \subset S$, AB denotes $\{ab \mid a \in A,\ b \in B\}$ and A^2 denotes $\{aa^1 \mid a,a^1 \in A\}$. A non-empty subset A of S is a submob if $A^2 \subset A$. Moreover, if A is a submob of S then $\overline{A}$ is also a submob of S. A subgroup of S is a subset G of S which is algebraically a group with its inherited multiplication. The multiplication in a subgroup G is clearly continuous, and, if G is locally compact, inversion is also continuous, so that G is a topological group. For each idempotent $e \in E(S)$, there is an unique maximal subgroup $H(e)$ containing e. If e and f are distinct idempotents, then the maximal subgroups $H(e)$ and $H(f)$ are disjoint. Moreover, if S is compact then $H(e)$ is closed for each $e \in E$. If S is not compact then this need not be the case. For example, let S be the interval $[0,\infty)$ with the usual topology and usual multiplication. Then S is not a compact mob and $H(1) = (0,\infty)$ is not closed in S.

A non-empty subset I of a mob S is called a left ideal if $SI \subset I$, a right ideal of $IS \subset I$ and an ideal if it is both left and right ideal. A minimal ideal of S is an ideal containing no other ideals. We denote the minimal ideal of S by $K(S)$. An ideal M of S is a maximal proper ideal of S if M is a proper ideal and for every ideal I of S such that $M \subset I$ we have that $I = M$ or $I = S$.

If a mob S has no proper ideal then S is said to be simple. For a compact mob S, $K(S)$ always exists and its structure is completely known; in particular, $K(S)$ is compact and is the disjoint union of a family of compact groups. This is a basic but very important result in the theory of compact mobs.

Let S be a compact mob, and let I be a closed ideal. Let $\frac{S}{I}$ denote the usual quotient space obtained by identifying all points of I, and let $\phi : S \to \frac{S}{I}$ be the natural map. In $\frac{S}{I}$, we define the multiplication by $\phi(x)\phi(y) = \phi(xy)$, and such a quotient space is called a Rees quotient. $\frac{S}{I}$ is a mob, because I is a closed ideal and the multiplication is well defined. Compactness of S and closure of I are used to prove $\frac{S}{I}$ Hausdorff and the multiplication is continuous. By using the Rees quotients, a disconnected mob can be made into a connected one if there is a closed ideal intersecting all components. For example, let W be a semigroup with two elements and $I = [0,1]$ be the usual thread. Then $W \times I$ is compact and has two components. $W \times 0$ is a closed ideal intersecting them all, and $\frac{W \times I}{W \times 0}$ is a connected fan as shown in Figure II.

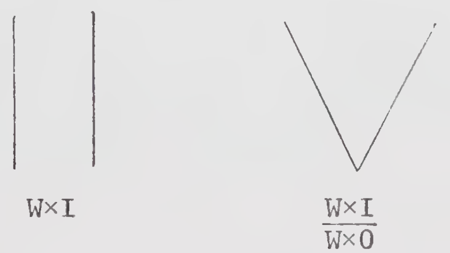


FIG. II

In topology, a homeomorphism is a function that is 1-1 onto, continuous, and whose inverse is a continuous function. In mobs, an iseomorphism is a function that is both isomorphism and homeomorphism. Thus, two mobs S and T are iseomorphis if there exists an iseomorphism from S onto T.

The following are the most frequently used facts in this dissertation.

(I) (Koch and Numakura). For $x \in S$, let $\Gamma_n(x) = \overline{\{x^p \mid p \geq n\}}$, $\Gamma(x) = \Gamma_1(x)$ and $K(x) = \cap\{\Gamma_n(x) \mid n \geq 1\}$. If $\Gamma(x)$ is compact, then $K(x)$ is an ideal of $\Gamma(x)$ and is a group. Thus, for an element x in a compact mob, the powers of x cluster at some idempotent, and, in particular, a compact mob contains an idempotent.

(II) (Koch and Wallace). If S is a compact mob and $\mathcal{U}$ is an open set in S containing an ideal of S, then $J_o(\mathcal{U})$, the maximal ideal contained in $\mathcal{U}$, is an open ideal of S. Moreover, any maximal proper ideal of S is open.

(III) (Faucett, Koch and Numakura). Let M be a maximal proper ideal of a compact mob S. Then $S-M$ is the disjoint union of compact groups and of compact sets A_α with the property $A_\alpha . A_\alpha \subset M$. (One of these types may fail to appear).

(IV) (Numakura). If S is compact, then each open prime ideal $P \neq S$, has the form $J_o(S-e)$, where e is non-minimal idempotent of S . If conversely e is a non-minimal idempotent, then $J_o(S-e)$ is an open prime ideal of S .

For more information about the theory of compact mobs, see J.M. Day [4], K.H. Hofmann and P.S. Mostert [11], A.B. Paalmann-de Miranda [21], and A.D. Wallace [29].

CHAPTER I

On algebraic radicals in mobs

We let A be any subset of a mob S. The algebraic radical of A is defined to be the set $\{x \in S \mid x^k \in A$ for some integer $k \geq 1\}$ and is denoted by $R(A)$. This set A is said to be radically stable if and only if $R(A) = R(\overline{A})$ holds. Obviously for any open subset A of S, A need not be radically stable. The purpose of this Chapter is to study some properties of the algebraic radicals of ideals in S. The main result is: Under some special conditions, any open ideal A of S can be radically stable without requiring that $R(A)$ be closed. Moreover, we will demonstrate that the notion of radical stability of an ideal in abelian mobs is useful : it gives a necessary and sufficient condition for the closure of a primary (prime) ideal to be primary (prime).

Throughout this Chapter, we use $\overline{C}$ to denote the closure of the set C, C' for the complement of C, and $B(C)$ for the boundery of C, and in possibly ambiguous situtations, the topological significance of a quantifier takes precedence over the alternative algebraic significance. Unless otherwise stated, S will be regarded as a compact mob with zero. The reader is referred to [21], [4] and [33] for terminology and notation.

§1. Preliminaries

In this section, pertinent notations, definitions and properties of algebraic radicals of an abelian mob S (not necessarily compact) will be given. Most of them are well known results in ring theory which will be used later.

Notation : Let A be a subset of S.

$J(A) = A \cup AS$, that is, the smallest ideal containing A.

$J_o(A)$ = the union of all ideals contained in A, that is, the largest ideal contained in A if there are any.

Definition 1.1 : (1) A mob S with zero is said to be 0-prime if whenever $a, b \in S$, $ab = 0$, then $a = 0$ or $b = 0$.

(2) A mob S is said to be an Ω-mob if for any two ideals I_1 and I_2 such that $I_1 \cap I_2 \neq \phi$, then either $I_1 \subset I_2$ or $I_2 \subset I_1$.

Definition 1.2 : (1) An ideal P of S is said to be prime if $ab \in P$ implies that $a \in P$ or $b \in P$.

(2) An ideal Q of S is said to be primary if $ab \in Q$ implies that $a \in Q$ or there exists an integer $k \geq 1$ such that $b^k \in Q$.

(3) An ideal R of S is said to be semi-prime if and only if $a^2 \in R$ implies that $a \in R$.

(4) Let A, B be ideals of S. Define $A : B = \{x \in S | xB \subset A\}$ and call it the ideal quotient of A and B. It is easy to see that $A : B$ is an ideal of S.

Definition 1.3 : (1) An ideal A is completely irreducible (irreducible) if and only if whenever A is the intersection of a family (finite family) of ideals, then A is a member of the family.

(2) An ideal A is w-reducible if A is the intersection of a family of open prime ideals containing A properly.

(3) An ideal A is strongly reducible (weakly reducible) if and only if A is the intersection of a finite family (infinite family) of ideals containing A properly.

Facts 1.4 : The algebraic radicals of S have the following properties:

Let A,B be any subsets of S. Then

(1) $A \subset R(A)$

(2) $A^k \subset B$ implies that $R(A) \subset R(B)$ for any $k \geq 1$.

(3) $R(R(A)) = R(A)$

If A,B are ideals of S, then

(4) $R(A)$ is an ideal of S

(5) $R(AB) = R(A \cap B) = R(A) \cap R(B)$

(6) If A is a primary ideal of S, then $R(A)$ is a prime ideal of S which is the smallest prime ideal containing A. Conversely, if $R(A)$ is a prime ideal of S, then A is primary.

(7) Let P, Q be ideals of S. Then Q is a primary ideal of S with $R(Q) = P$ if and only if

(i) $Q \subset P \subset R(Q)$

and (ii) $ab \in Q$, $a \notin Q$ imply that $b \in P$.

(8) If A is an open subset of S, then $R(A)$ is open.

(9) If M is a non-prime maximal proper ideal of S, then $\mathcal{R}(M) = S$. If M is any ideal of S such that $\mathcal{R}(M) = S$, then M is primary.

Remark : Some of the above facts are not true if S is not an abelian mob. For example, let $S = \{0,1,2,3,4\}$ and define the multiplication on S by the following table:

.	0	1	2	3	4
0	0	0	0	0	0
1	0	1	3	3	1
2	0	4	0	2	0
3	0	1	0	3	0
4	0	4	2	2	4

Associativity of the multiplication can be easily verified to confirm that S is a non-abelian mob. $A = \{0\}$ is an ideal of S, but $\mathcal{R}(A) = \{0,2\}$ is not an ideal of S. Moreover, $B = \{0,1,3\}$, $C = \{0,2,4\}$ are ideals of S. But $\mathcal{R}(B) = \{0,1,2,3\}$, $\mathcal{R}(C) = \{0,2,4\}$, $B \cdot C = \{0,2,3,4\}$, $\mathcal{R}(B \cap C) = \mathcal{R}(0) = \{0,2\}$, $\mathcal{R}(BC) = \{0,2,3,4\}$, $\mathcal{R}(B) \cap \mathcal{R}(C) = \{0,2\}$. Hence $\mathcal{R}(BC) \neq \mathcal{R}(B) \cap \mathcal{R}(C)$, and $\mathcal{R}(BC) \neq \mathcal{R}(B \cap C)$.

The proofs of some of the above results are analogous to those in ring theory and can be found in [33]. The proofs of (7) - (8) are proved here for the convenience of the reader.

Proof : (6) Let $xy \in \mathcal{R}(A)$, then there exists an integer $k \geq 1$ such that $(xy)^k \in A$, that is $x^k y^k \in A$. If $x^k \in A$, then $x \in \mathcal{R}(A)$.

If $x^k \notin A$, then there exists an integer $m \geq 1$ such that $(y^k)^m \in A$ since A is primary. Hence $y \in R(A)$. So $R(A)$ is a prime ideal of S. Suppose $P_1 \supset A$ is a prime ideal. Let $x \in R(A)$, so there exists an integer $k \geq 1$ such that $x^k \in A \subset P_1$. As P_1 is prime, we have $x \in P_1$. That is $R(A) \subset P_1$, so $R(A)$ is the smallest prime ideal containing A. The other part of $|6|$ is easy to see.

Remark : Unless A is primary already, there does not exist a smallest prime ideal containing A.

(7) Suppose $R(Q) = P$, then clearly $Q \subset P \subset R(Q)$. And if $ab \in Q$, $a \notin Q$, then, since Q is a primary ideal of S, there exists an integer $k \geq 1$ such that $b^k \in Q$, so $b \in R(Q) = P$.

Conversely, let $ab \in Q$, $a \notin Q$, then, by (ii), $b \in P = R(Q)$. That is, $b^k \in Q$ for some integer $k \geq 1$. Hence Q is primary. By (1), we have $P \subset R(Q)$. In order to prove $P = R(Q)$, we only need to show that $R(Q) \subset P$. Let $x \in R(Q)$, then there exists an integer $k \geq 1$ such that $x^k \in Q$. Suppose k is the smallest such integer. If $k = 1$, $x \in Q$, so $x \in P$. If $k > 1$, $x \cdot x^{k-1} = x^k \in Q$, so $x^{k-1} \notin Q$, by (ii), $x \in P$. Our proof is complete.

(8) Take $x \in R(A)$, then there is an integer $k \geq 1$ such that $x^k \in A$. Let $\mu = \prod_1^k S \to S$ defined by $\mu(x_1, x_2, \ldots, x_k) = x_1 x_2 \ldots x_k$. Then μ is a continuous map and $\mu^{-1}(A)$ is an open neighborhood of $x \ldots x$ (k terms). Let V be an open neighborhood of x such that $V \times V \times \ldots \times V$ (k terms) $\subseteq \mu^{-1}(A)$.

Hence $\mu(V \times V \times \ldots \times V) = V \cdot V \ldots V$ (k terms) $\subset \mu\mu^{-1}(A) = A$. That is $V^k \subset A$. This implies $V \subset \mathcal{R}(A)$, so $\mathcal{R}(A)$ is open.

§2 Prime and Primary ideals

We are going to study, in this section, the prime and primary ideals of S , and, in particular, the algebraic radical of such ideals and their relationships.

Proposition 2.1 : An ideal A of S is a compact prime ideal if and only if $\frac{S}{A}$ is a 0-prime mob.

Proof : Suppose A is a compact prime ideal of S . Then A is closed in S . The Rees quotient $\frac{S}{A}$ is formed by shrinking A to a single point with the quotient topology. $\frac{S}{A}$ is a mob. Recall that the multiplication $*$ of $\frac{S}{A}$ is defined in the following way:

$$a*b = ab \quad \text{if} \quad a,b \quad \text{and} \quad ab \quad \text{are in} \quad S - A .$$

$$a*b = 0 \quad \text{if} \quad ab \in A$$

$$a*b = 0 \quad \text{if} \quad a = 0 \quad \text{or} \quad b = 0$$

If $a*b = 0$, there are two possible cases: Either (i) $a = 0$ or $b = 0$, or (ii) $ab \in A$. In case (ii) , since A is prime, we have $a \in A$ or $b \in A$. This implies that $a = 0$ or $b = 0$ in $\frac{S}{A}$. Thus in either case $a = 0$ or $b = 0$. Hence $\frac{S}{A}$ is 0-prime. Conversely, assuming that $\frac{S}{A}$ is an 0-prime mob, since $\frac{S}{A}$ is Hausdorff, the ideal A is closed in S and hence is compact. Suppose $x * y = 0$ in $\frac{S}{A}$, then we have $x = 0$ or $y = 0$ in $\frac{S}{A}$.

This means that $x \in A$ or $y \in A$ in the mob S . Hence A is a compact prime ideal of S .

Theorem 2.2 Let A be an ideal of S such that $R(A)$ is proper maximal in S . Then A is primary if and only if $\frac{S}{R(A)}$ is an abstract completely 0-simple semigroup.

Proof : Suppose A is a primary ideal of S , then $R(A)$ is a prime ideal. As S is compact, it follows that $R(A)$ is open [21], page 28. By theorem 2 of [20], page 677, $R(A)$ has the form $J_o(S-e)$ with e being a non-minimal idempotent of S . Therefore there exists $e^2 = e \notin R(A)$. Now form the Rees quotient $\frac{S}{R(A)}$. Clearly , $\frac{S}{R(A)}$ is an abstract 0-simple semigroup [21], page 59, and contains e . Hence by [6], page 655, $\frac{S}{R(A)}$ is completely 0-simple. Conversely, suppose that $\frac{S}{R(A)}$ is completely 0-simple. Then there exists an $e^2 = e \notin R(A)$. Clearly, e is non-minimal. By the maximality of $R(A)$, we have $R(A) = J_o(S-e)$. By theorem 2 of [20], page 679, again, $R(A)$ is an open prime ideal of S . Now take $xy \in A$, then $xy \in R(A)$. Thus $x \in R(A)$ or $y \in R(A)$. This implies that A is primary.

Corollary. If E , the set of idempotents of S , is contained in a maximal proper ideal J of S , then J is a primary ideal of S .

Proof : By [10], page 655, $\frac{S}{J}$ is either the zero semigroup of order two or else completely 0-simple.

Since $E \subset J$, $\frac{S}{J}$ contains no idempotents other than zero and hence $\frac{S}{J}$ is the zero semigroup of order two. Suppose $xy \in J$, $x \notin J$, $y \notin J$. Then $x \in S-J$, $y \in S-J$ in $\frac{S}{J}$. Since $\frac{S}{J}$ is the zero semigroup of order two, we have $y^2 = 0$, $x^2 = 0$ in $\frac{S}{J}$. This implies that $x^2 \in J$, $y^2 \in J$ in the mob S. Thus J is a primary ideal of S.

<u>Theorem 2.3</u> : Let S be a connected mob with unit and let J be an open ideal of S such that $\mathcal{R}(J)$ is proper in S. Then J is a primary ideal if and only if the boundary of $\mathcal{R}(J)$ is a subset of S.

<u>Proof</u> : If J is an open primary ideal, then by 1.4(9), $\mathcal{R}(J)$ is an open prime ideal of S. Clearly, $\mathcal{R}(J)^2 \subset \mathcal{R}(J)$ and $(S-\mathcal{R}(J))^2 \subset S - \mathcal{R}(J)$. Thus $(B(\mathcal{R}(J)))^2 \subset B(\mathcal{R}(J))$. Conversely, suppose $\mathcal{R}(J)$ is not prime, take $a,b \in S-\mathcal{R}(J)$ with $ab = ba \in \mathcal{R}(J)$. Then $Sa \cap \mathcal{R}(J) \neq \phi$ and $Sa \cap (S-\mathcal{R}(J)) \neq \phi$ so, it follows that $Sa \cap B(\mathcal{R}(J) \neq \phi$. Similarly, $bS \cap B(\mathcal{R}(J)) \neq \phi$. Since $\mathcal{R}(J)$ is open and S is connected, we have $\phi \neq [Sa \cap B(\mathcal{R}(J))][bS \cap B(\mathcal{R}(J))] \subset (Sa \cdot bS) \cap (B(\mathcal{R}(J)))^2 \subset \mathcal{R}(J) \cap B(\mathcal{R}(J)) \neq \phi$, for $B(\mathcal{R}(J))$ is a subset of S. This contradiction proves that $\mathcal{R}(J)$ is prime, which implies that J is a primary ideal of S.

<u>Corollary</u> : Let S be a connected mob and J be an open ideal of S. Then $S-\mathcal{R}(J)$ is a submob of S if and only if the boundary of $\mathcal{R}(J)$ is a submob of S.

A.D. Wallace has proved the following result: Let S be an abelian mob (not necessarily abelian). Then each open prime ideal is completely irreducible, and each completely irreducible ideal is open [29], page 39. One would naturally ask whether the irreducibility of an ideal Q in an abelian semigroup is a necessary and sufficient condition for Q to be primary. (This question was asked by A.D. Wallace in his lecture notes on topological semigroups, problem J6 page 39 of [29]). We show here, by giving a counter example, that the answer to this question is negative.

Example 2.4. Let S be an abelian semigroup consisting of four elements $\{0,a,b,c\}$ with multiplication table:

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	b
c	0	0	b	b

The sets $\{0,b\}$, $\{0,b,c\}$, $\{0,a,b\}$ are ideals of S . Now $\{0,b\} = \{0,b,c\} \cap \{0,a,b\}$. It is easily seen that $\{0,b\}$ is a primary ideal of S , but is not irreducible. Thus we have shown that primary ideals in abelian mobs are not necessarily irreducible.

Theorem 2.5 : If Q is an open semi-prime ideal of S , then Q is prime or w-reducible

Before proving this theorem, we need the following two lemmas.

Lemma 2.6 . Q is a semi-prime ideal if and only if $\mathcal{R}(Q) = Q$.

Proof : If $\mathcal{R}(Q) = Q$, then it is easily seen that Q is semi-prime. Conversely, suppose that $Q \underset{\neq}{\subset} \mathcal{R}(Q)$, then there exists $a \in \mathcal{R}(Q)$ with $a \notin Q$. Let $k > 1$ be the minimal integer such that $a^k \in Q$. Suppose Q is semi-prime. Then k must be odd. Write $k = 2n + 1$ $(n > 0)$. Since Q is an ideal, we infer that $a^{k+1} = a^k \cdot a \in Q$. Thus $a^{k+1} = a^{2n+2} = (a^{n+1})^2 \in Q$. Since Q is semi-prime, it follows that $a^{n+1} \in Q$. This contradicts the minimality of k . Hence $\mathcal{R}(Q) = Q$.

Lemma 2.7 . Let Q be an open ideal of S , then $\mathcal{R}(Q) = \bigcap_\alpha P_\alpha$, where $\{P_\alpha\}$ are all the open prime ideals of S containg Q .

Proof : Take $x \in \mathcal{R}(Q)$. Then there exists integer $k \geq 1$ such that $x^k \in Q \subseteq P_\alpha$ for all α . Since P_α is prime, $x \in P_\alpha$ for all α , that is, $x \in \bigcap_\alpha P_\alpha$. Hence $\mathcal{R}(Q) \subset \bigcap_\alpha P_\alpha$. Conversely, supposing that $\bigcap_\alpha P_\alpha \not\subset \mathcal{R}(Q)$. Then we can find an element y of $\bigcap_\alpha P_\alpha$ such that $y \notin \mathcal{R}(Q)$. We have $\overline{\{y, y^2 \ldots\}} = \Gamma(y) \subset J(y) \subset \bigcap_\alpha P_\alpha$.

Since $\Gamma(y)$ is compact and Q is open, we can prove that there exists an idempotent e such that $e \in \Gamma(y) \subset \bigcap_\alpha P_\alpha$ and $e \notin Q$. Thus $J_o(S-e) \supset Q$. By theorem 2 of [20], page 677, $J_o(S-e)$ is an open prime ideal of S . Therefore $J_o(S-e) \supset \bigcap_\alpha P_\alpha$. This implies that $e \notin \bigcap_\alpha P_\alpha$, a contradiction. Thus $\bigcap_\alpha P_\alpha \subset \mathcal{R}(Q)$.

By now, one can easily see that theorem 2.5 is an immediate consequence of these lemmas.

Corollary 1 : (i) If A is an ideal, then $R(A)$ is a semi-prime ideal.

(ii) A is an open semi-prime ideal of S if and only if $A = \cap\{P_\alpha | P_\alpha$ is an open prime ideal containing $A\}$.

Corollary 2 : Let Q be an open semi-prime ideal of S . If B is an ideal of S which is not contained in Q , then B contains an idempotent e with $eS \not\subset Q$.

Proof : Let $b \in B - Q$. Consider the principal ideal $J(b)$ generated by b . Clearly $J(b)$ is compact, $J(b) \subset B$, $J(b) \not\subset Q$. Now let M be the collection of all compact ideals $\{J_i\}_{i \in I}$ with the properties $J_i \subset B$, $J_i \not\subset Q$. By the same arguments as lemma 8 [20], page 676, we prove that there exists a minimal member J in M with $J \subset B$, $J \not\subset Q$. Now let $x \in J - Q$ claim $x^2 J \not\subset Q$. Suppose $x^2 J \subset Q$. Since Q is semi-prime, by lemma 2.6 and lemma 2.7, $Q = \bigcap_\alpha P_\alpha$, where P_α are open prime ideals containing Q . As S is abelian, we have $J(x)^3 \subset x^2 J \subset Q \subset P_\alpha$ for all α . This implies that $J(x) \subset P_\alpha$ for all α . Hence $J(x) \subset \bigcap_\alpha P_\alpha = Q$, a contradiction. So we assert that $x^2 J \not\subset Q$. Since $x^2 J \subset J$ and J is minimal, we have $x^2 J = J$. Now $x \in J \Rightarrow \Gamma(x) \subset J \Rightarrow$ there is an idempotent $e \in J \Rightarrow eS \subset J \Rightarrow$ $eS = J$ since J is a minimal ideal and $eS \not\subset Q$.

Therefore $eS = J = x^2J \nsubseteq Q$. Our proof is complete.

Theorem 2.8 : Let S be a compact mob such that $S^2 = S$. If A is the intersection of maximal proper ideals of S , then A is either prime or w-reducible, and $S-A$ is the disjoint union of compact groups.

Proof : Let $\mathcal{M}$ be the family of maximal proper ideals of S . By P.A. Grillet [9], page 503, every maximal ideal of S is prime if and only if $S = S^2$. S is compact, so each maximal proper ideal of S is an open prime ideal of S . This implies A is w-reducible. Obviously, $S-A = \cup\{S-M | M \in \mathcal{M}\}$. We claim that the complement of distinct maximal proper ideals are disjoint. For suppose that M_1 and M_2 are distinct maximal proper ideals such that $(S-M_1) \cap (S-M_2) \neq \phi$. Then we not that $M_3 = M_1 \cup M_2$ is consequently a proper ideal of S , and M_3 properly contains M_1 and M_2 which is a contradiction. By a result of Faucett-Koch-Numakura [6], page 656, we have that $S-M$ is the disjoint union of compact groups. Thus $S-A$ is indeed the disjoint union of compact groups.

Theorem 2.9 : Let F be a closed ideal of S and let $\mathcal{G} = \{$open ideals G_α of $S | G_\alpha \supset F\}$. Then $F = \bigcap_\alpha G_\alpha$, $G_\alpha \in \mathcal{G}$ for all α . In otherwords, F is weakly reducible if the family $\mathcal{G}$ exists.

Proof : Trivially $F \subset \bigcap_\alpha G_\alpha$. To prove the converse containment, we only need to show that for any element $x \nsubseteq F$, $x \nsubseteq \bigcap_\alpha G_\alpha$.

Since F is closed in S, it is compact. As S is compact Hausdorff, it is a regular space and hence there exists an open neighborhood V containing F but excluding x. By the compactness of S, we have that $J_o(V)$ is an open ideal of S. Obviously, $F \subset J_o(V)$. Hence $J_o(V) \in \mathcal{G}$. Clearly $x \notin J_o(V)$. This implies that $x \notin \bigcap_\alpha G_\alpha$.

Corollary : If S satisfies the second axiom of countability, then F is a G_δ-ideal, that is, F can be expressed as a countable intersection of open ideals containing F.

This is because compact and T_2 imply regular, and regular and second countability imply metriziable and every closed set in any metric space is G_δ.

Remark : The author learned from Professor J. M. Day that this theorem is due to A.D. Wallace in a form like this: "Every neighborhood of a closed ideal F contains an open ideal containing F". It appeared in Wallace's notes from University of Florida 1964-1965.

Theorem 2.10 : Let S be an abelian mob (not necessarily compact). If the algebraic radical of an ideal A is non-prime, then it is strongly reducible.

Proof : Since $\mathcal{R}(A)$ is not prime, we can find elements x,y in S such that $xy \in \mathcal{R}(A)$ but $x \notin \mathcal{R}(A)$, $y \notin \mathcal{R}(A)$. Consider $\mathcal{R}(A) : J(y) = \{z \in S \mid zJ(y) \subset \mathcal{R}(A)\}$.

Then $R(A) = J(y)$ is an ideal of S with $R(A) \subset R(A) = J(y)$. We claim that $R(A) \neq R(A) : J(y)$. In fact since $xy \in R(A)$, we have that $xJ(y) = x(\{y\} \cup yS) = \{xy\} \cup xyS \subset R(A)$. Thus $x \in R(A) : J(y)$ but $x \notin R(A)$. Now clearly $R(A) \subset (R(A) \cup J(y)) \cap (R(A) : J(y))$. On the other hand, if $t \in (R(A) \cup J(y) \cap (R(A) : J(y))$ but $t \notin R(A)$, then we must have $t \in J(y)$. Hence $t^2 \in tJ(y) \subset R(A)$. Since $R(A)$ is semi-prime, we have $t \in R(A)$. Hence we have shown that $R(A) = (R(A) \cup J(y)) \cap (R(A) : J(y))$ and hence $R(A)$ is strongly reducible.

Corollary 1 : Let F be a closed ideal which is strongly reducible with respect to a finite family of open primary ideals, then $R(F)$ is the intersection of a finite number of open prime ideals of S .

Proof : $F = \bigcap_{i=1}^{n} Q_i$, with Q_i open primary ideal of S . So $F \subset Q_i \subset R(Q_i) = P_i$ for all i , then $F \subset \bigcap_{i=1}^{n} P_i$. Thus

$$R(F) \subset R(\bigcap_i P_i) = \bigcap_{i=1}^{n} R(P_i)$$

$$= \bigcap_{i}^{n} P_i \ .$$

Let $x \in \bigcap_i^n P_i$, then $x \in P_i$ for all $i = 1,\ldots,n$, which implies $x \in R(Q_i)$. Hence for each i , there exists an integer $k_i \geq 1$ such that $x^{k_i} \in Q_i$.

Take $\beta = \max\{k_i\}$, $i = 1,2,\ldots,n$, then we have $x^\beta \in \bigcap_{i=1}^{n} Q_i$ which implies $x \in R(\bigcap_{i=1}^{n} Q_i) = R(F)$. By facts 1.4(8), we have shown that $R(F)$ is the intersection of a finite number of open prime ideals of S .

<u>Corollary 2</u> : If $Q_1,\ldots,Q_n$ is a finite number of primary ideals of S with the same algebraic radical C , then $Q = Q_1 \cap Q_2 \ldots \cap Q_2$ is primary and has the algebraic radical C .

<u>Corollary 3</u> : Let Q be an open primary ideal of the compact mob S with $R(Q) = P$. If A is any closed ideal of S with $A \not\subset Q$, then $Q : A$ is an open primary ideal of S with $R(Q : A) = P$.

<u>Proof</u> : Since Q is open, Q' is closed and hence compact. A is also compact. If $x \in Q : A$, then $xA \cap Q' = \phi$. By the continuity of multiplication and the compactness of A , there exists an neighborhood V of x such that $VA \cap Q' = \phi$. That is $VA \subset Q$. Hence $x \in V \subset Q : A$, that is, $Q : A$ is open. By 1.4(7) and the fact that $(Q : A)A \subset Q$, we can obtain that (i) $Q : A \subset P \subset R(Q : A)$ and (ii) $ab \in Q : A$, $a \not\in Q : A$ imply that $b \in R(Q : A)$. Hence, by 1.4(7) again, $Q : A$ is an open primary ideal of S with $R(Q : A) = P$.

<u>Corollary 4</u> : If Q is a compact primary ideal of the compact mob S with $R(Q) = P$ and if A is any ideal $\not\subset Q$, then $Q : A$ is a compact primary ideal of S with $R(Q : A) = P$.

In what follows, if the algebraic radical of an ideal A is an open primary ideal P, then A is called a P-ideal of S.

The following result, which is a simple consequence of a theorem of W.M. Faucett [5], page 749, gives a condition when and ideal of a compact connected mob S can be a P-ideal.

Proposition 2.11 : Let S be a compact connected mob with unit and A be an ideal of S. Let z be a cut point of S which cuts S at $R(A)$, that is $S - \{z\} = R(A) \cup B$, $R(A)|B$ then A is a P-ideal if and only if z is an idempotent.

Proof : See W.M. Faucett, Theorem 1 [5], page 749.

Prosposition 2.12 : Let S be a compact connected mob and P be a prime ideal of S. Then P is connected if P is contained in the intersection of all maximal proper ideals of S.

Proof : Let A be the intersection of all maximal proper ideals of S. By Koch and Wallace [14], page 683, we have $A \subset S^2$. Hence $P \subset S^2$. Now let C be a component of P and let $x \in P$, then $x = ab$. As P is prime, then say, $a \in P$. Now $x \in aS \subset P$ and aS meets C since S has a zero, so $x \in aS \subset C$. Thus we obtain that $P = C$, P is connected.

Proposition 2.13 : The set of all P-ideals of S forms a filter on S .

This proposition follows immediately by observing that

(1) Any finite intersection of P-ideals of S is a P-ideal.

(2) Any arbitrary union of P-ideals of S is still a P-ideal.

Moreover, we remark that this union is a subset of S and is an open prime ideal of S .

Now, let e be an idempotent of a compact mob S . We say that an element $x \in S$ belongs to the idempotent e if e is the unique idempotent of $\Gamma(x) = \overline{\{x,x^2,\ldots\}}$. Let us denote by $B_\alpha = \{x \in S | e_\alpha \in \Gamma(x)\}$. We shall call it a B-class. St. Schwarz [21], page 119, has proved that any compact abelian mob S can be written as the union of disjoint B-classes.

Theorem 2.14 : Let A be a P-ideal of S . Then there exists at least one B-class which meets A but is disjoint from $S - R(A)$.

Proof : We may assume that there exists a B-class B_{α_o} such that $B_{\alpha_o} \cap A \neq \phi$. Let $x \in B_{\alpha_o} \cap A$. Then $x \in A$ and $x \in B_{\alpha_o}$. Consider the principal ideal $J(x)$ generated by x . Clearly $J(x)$ is compact and $\{x,x^2,\ldots\} \subset J(x) \subset A$. Thus $\Gamma(x) \subset J(x)$. $\Gamma(x)$ has a unique idempotent which must be e_{α_o} since $x \in B_{\alpha_o}$. Now, suppose there exists an element $y \in B_{\alpha_o} \cap (S-R(A))$.

The element y also belongs to the idempotent e_{α_0} . But, since $y \in S-\mathcal{R}(A)$, and $\mathcal{R}(A)$ is prime, we have $\{y,y^2,\ldots\} \subset S-\mathcal{R}(A)$. As $\mathcal{R}(A)$ is open, $S-\mathcal{R}(A)$ is compact in S . It follows that $\overline{\{y,y^2,\ldots\}} = \Gamma(y) \subset S-\mathcal{R}(A)$. Therefore $e_{\alpha_0} \in \Gamma(y) \subset S-\mathcal{R}(A)$. Therefore, $e_{\alpha_0} \in \Gamma(y) \subset S-\mathcal{R}(A)$. This is impossible since A and $S-\mathcal{R}(A)$ are disjoint. Hence $B_{\alpha_0} \cap \subset S-\mathcal{R}(A)) = \phi$.

<u>Corollary.</u> Any P-ideal A contains exactly the same number of disjoint B-classes as $\mathcal{R}(A)$. More precisely, $A \cap \{\underset{\alpha}{\cup} B_\alpha\} = \underset{\alpha}{\cup} (A \cap B_\alpha)$ with $B_\alpha \subset P$.

§3. <u>Stability of algebraic radicals.</u>

<u>Proposition 3.1.</u> If A is a subset of S with $\mathcal{R}(A)$ closed, and $x \in S$ is such that $A \subset xA$, then we have $\mathcal{R}(A) = \mathcal{R}(xA)$. In other words, the closed algebraic radical of A would not be expanded under any translation.

<u>Proof</u> : By the "Swelling lemma" [11], page 15, $A \subset xA \subset \overline{A}$. Hence $\mathcal{R}(A) \subset \mathcal{R}(xA) \subset \mathcal{R}(\overline{A})$. We only need to prove that $\mathcal{R}(\overline{A}) \subset \mathcal{R}(A)$. Since $A \subset \mathcal{R}(A)$, we have $\overline{A} \subset \overline{\mathcal{R}(A)} = \mathcal{R}(A)$. Consequently, $\mathcal{R}(\overline{A}) \subset \mathcal{R}(\mathcal{R}(A)) = \mathcal{R}(A)$. Thus we have obtained that $\mathcal{R}(xA) = \mathcal{R}(A)$.

<u>Theorem 3.2.</u> (Main theorem). Let A be an open ideal of S . Then A is radically stable if and only if $\mathcal{R}(\overline{A})$ does not contain any idempotent lying outside A .

In order to prove this theorem, the following lemma is crucial.

Lemma 3.3. Let A be any open ideal of S. If B is an ideal which is not contained in $R(A)$, then B has an idempotent not in A.

Proof : Since A is an ideal of S, so is $R(A)$. As $B \not\subset R(A)$, there exists an element $b \in B$ such that $b \notin R(A)$. Now $J(b) = \{b\} \cup bS \subset B$, and $J(b)$ is compact, for S is compact. So there exists an idempotent $e^2 = e \in \Gamma(b) \subset J(b) \subset B$. Suppose on the contrary that $e \in A$. Then $K(b) = e\,\Gamma(b) \subset A$ where $K(b) = \bigcap_{n=1}^{\infty} \overline{\{b^i \mid i \geq n\}}$ [21], page 25. Since A is open, we must have $b^n \in A$ for some integer $n \geq 1$. For otherwise, suppose $b^n \notin A$ for all integer $n \geq 1$. Then $b^n \in A'$ for all integer $n \geq 1$. Because A' is closed and hence compact, we have $\Gamma(b) \subset A'$, which implies $e \notin A$, a contradiction. Thus $b^n \in A$ implies $b \in R(A)$, which is impossible.

Remark : For any compact abelian mob S and A a non-empty open subset of S, if B is a submob of S such that $B \not\subset R(A)$, then $\overline{B}$ contains an idempotent which is not in $R(A)$.

We are now ready to prove theorem 3.2. As A is an ideal, so is $\overline{A}$ and $R(\overline{A})$. For the necessity, we suppose that $R(\overline{A}) \not\subset R(A)$. Then, by our lemma 3.3, there exists an idempotent $e^2 = e \in R(\overline{A})$, $e \notin A$.

But we assume that such idempotent does not exist. Hence, $\mathcal{R}(\overline{A}) \subset \mathcal{R}(A)$. As $\mathcal{R}(A) \subset \mathcal{R}(\overline{A})$ always holds, we have $\mathcal{R}(\overline{A}) = \mathcal{R}(A)$ that is, A is radically stable. For the converse part, we assume that A is radically stable, that is, $\mathcal{R}(\overline{A}) = \mathcal{R}(A)$. Suppose there exists $e^2 = e \in \mathcal{R}(\overline{A})$. Then $e \in \mathcal{R}(A)$, so there exists $k \geq 1$ such that $e^k \in A$. Thus $e \in A$ and hence, $\mathcal{R}(\overline{A})$ contains no idempotents which are not in A . Our proof is complete.

<u>Corollary 1</u> : Let A be a proper ideal of the mob S . Then any ideal of S properly containing $\mathcal{R}(A)$ contains a compact group which is disjoint from A . Conversely, let G be a compact group in S such that G is disjoint from an open ideal A , and suppose that A contains all the other idempotents of S . Then $\mathcal{R}(A)$ is an open ideal of S disjoint from G .

<u>Proof</u> : By corollary 2 of lemma 2.7, we have $e S e \not\subset \mathcal{R}(A)$ for some idempotent e . Now $e S e$ is a compact submob of S with identity e . Consider $G_e = \{g \in e S e \mid gg^{-1} = e\}$. This is the maximal subgroup of $e S e$. It is known that G_e is a compact subgroup of $e S e$ [12], page 13. We claim that $e \notin \mathcal{R}(A)$. For if $e \in \mathcal{R}(A)$, then $e S e \subset \mathcal{R}(A)$, a contradiction. Let us now suppose that $G_e \cap \mathcal{R}(A) \neq \phi$, then there exists $g \in G_e$ such that $g \in \mathcal{R}(A)$. Since $\mathcal{R}(A)$ is an ideal of S , $gg^{-1} = e \in \mathcal{R}(A)$, which is impossible. For the converse part, suppose $G \cap A = \phi$. Since G is a group , $g^k \in G$ for all $k \geq 1$ which $g \in G$.

Hence $g^k \notin A$ for all $k \geq 1$. This implies $g \notin \mathcal{R}(A)$. Thus $G \cap \mathcal{R}(A) = \phi$. As G and S are compact, $J_o(S-G)$ is an open ideal of S. Clearly, $\mathcal{R}(A) \subset J_o(S-G)$. Suppose that $J_o(S-G) \notin \mathcal{R}(A)$. Then by our lemma 3.3, there exists $e^2 = e \in J_o(S-G)$, $e \notin A$. This contradicst our assumption on A. Hence $\mathcal{R}(A) = J_o(S-G)$ and hence $\mathcal{R}(A)$ is an open ideal of S.

<u>Corollary 2</u> : Let S be an Ω-mob. If A is an open ideal of S which is not radically stable and $\overline{\mathcal{R}(A)}$ is semi-prime, then $\mathcal{R}(\overline{A})$ is closed and has the form eS with $e^2 = e \notin \mathcal{R}(A)$.

<u>Proof</u> : The non-radical stability of A implies that $\mathcal{R}(A) \subsetneq \mathcal{R}(\overline{A})$. By using the same method as lemma 8 in [20], page 676, and our lemma 3.3, we can prove that there exists a minimal closed ideal M contained in $\mathcal{R}(\overline{A})$, but not contained in $\mathcal{R}(A)$. Moreover M has the form eS with $e^2 = e \notin \mathcal{R}(A)$. Since S is compact, $eS \cap \mathcal{R}(A) \neq \phi$. As S is Ω, it follows that $\mathcal{R}(A) \subset eS \subset \mathcal{R}(\overline{A})$. Hence $\overline{\mathcal{R}(A)} \subset eS$. Since $\overline{\mathcal{R}(A)}$ is semi-prime, we have $\mathcal{R}(\overline{\mathcal{R}(A)}) = \overline{\mathcal{R}(A)}$. Thus $\overline{A} \subset \overline{\mathcal{R}(A)}$ implies that $\mathcal{R}(\overline{A}) \subset \overline{\mathcal{R}(A)}$. We have, therefore, $\mathcal{R}(\overline{A}) = eS$ with $e^2 = e \notin A$.

<u>Corollary 3</u> : Let A be an ideal which is radially stable in S. Then A is a primary ideal if and only if $\overline{A}$ is a primary ideal.

<u>Proof</u> : We only need to observe that an ideal A is primary if and only if $\mathcal{R}(A)$ is prime.

Here we give two examples to demonstrate that, without radical stability, the closure of a prime (Primary) ideal need not be prime (Primary).

Example 3.3. Let S be the subset of the plane defined by $S = ([0,1] \times 0) \cup (1 \times [-1,1]))$ (see Figure I.) where the underlined brackets denote the intervals, and define a multiplication on S by:

$\langle x,0\rangle * \langle 1,v\rangle = \langle x,0\rangle$ for all points $x \in [a,b]$, $v \in [c,d]$.

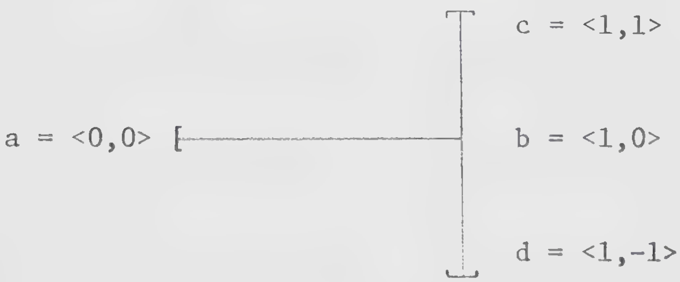
$\langle x,0\rangle * \langle y,0\rangle = \langle xy,0\rangle$ for all points $x,y \in [a,b]$.

$\langle 1,x\rangle * \langle 1,y\rangle = \langle 1,xy\rangle$ for all points $x,y \in [b,c]$.

$\langle 1,x\rangle * \langle 1,y\rangle = \langle 1,-xy\rangle$ for all points $x,y \in [b,d]$.

$\langle 1,x\rangle * \langle 1,y\rangle = \langle 1,0\rangle$ if $x \in [b,d]$, $y \in [b,c]$ and vice versa.

Where xy is the usual product of x and y .



(Figure (I.)

Clearly, $[a,b)$ is a prime ideal of S . Also $\langle 1,1\rangle * \langle 1,-1\rangle = \langle 1,0\rangle \in [a,b]$, but $\langle 1,1\rangle, \langle 1,-1\rangle$ are note points in $[a,b]$. Hence, the closure of $[a,b]$ is not a prime ideal of S .

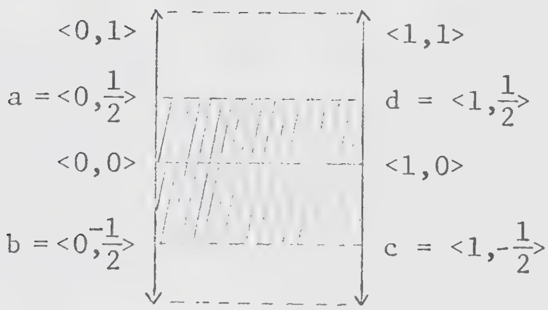
Example 3.4. Let S be the subset of the plane define by $S = ([0,1] \times (-1,1)) \cup (1 \times [1,-1])$ (see Figure II.) where the

underlined brackets denote intervals, and define multiplication on S by:

$\langle x,y\rangle * \langle u,v\rangle = \langle xu,yv\rangle$ for all points $\langle x,y\rangle$, $\langle u,v\rangle$ in the upper half plane.

$\langle x,y\rangle * \langle u,v\rangle = \langle xu,-yv\rangle$ for all points $\langle x,y\rangle$, $\langle u,v\rangle$ in the lower half plane.

$\langle x,y\rangle * \langle u,v\rangle = \langle xu,0\rangle$ if one of the points lies in the upper half plane and the other lies in the lower half plane.



(Figure II.)

Clearly, the rectangle $Q = \underline{(0,1)} \times \underline{(\frac{-1}{2},\frac{1}{2})}$ is a primary ideal of S , but the closure of Q is not primary.

Remark 1 : Every ideal of the usual thread I is a primary ideal. By a usual thread, we mean a mob iseomorphic to [0,1] with its usual topology and usual real multiplication. Obviously, the minimal ideal $\{0\}$, of I is primary. Any non-minimal ideal of I has the form $[0,x)$ or $[0,x]$ for a fixed x in $(0,1)$ [21], page 84. To see that $[0,x]$ is primary, suppose $ab \in [0,x)$, $a \notin [0,x)$.

Then $0 \le ab < x$, $x \le a \le 1$. Hence, $0 \le b < \frac{x}{a}$, $\frac{x}{a} \le 1$. Thus $0 \le b < 1$. Since x is fixed, there exists $k \ge 1$ such that $b^k < x$. As $[0,x)$ is radically stable, $[0,x]$ is also a primary ideal of I .

Remark 2 : Every ideal of the min-thread I is prime. By a min-thread, we mean a mob iseomorphic to $[0,1]$ with multiplication $x * y = \min \{x,y\}$. This remark is clear.

§4. On the boundary of algebraic radicals.

In this section, we prove two very important theorems. These two theorems tell us that: In a mob S , the boundary of the algebraic radical of an open ideal A is closely related with a compact topological group in S .

Theorem 4.1. Let S be a compact mob and A be an open ideal of S . Then either $B(\mathcal{R}(A)) = \phi$ or there exists a non-zero compact group which lies in the boundary of $\mathcal{R}(A)$.

Proof : Since A is an open ideal of S , by 1.4(8), $\mathcal{R}(A)$ is also an open ideal of S . If $\mathcal{R}(A) = S$, then clearly $B(\mathcal{R}(A)) = \phi$. Suppose that $B(\mathcal{R}(A)) = \phi$, then $\overline{\mathcal{R}(A)} - \mathcal{R}(A) \neq \phi$. Since $\overline{\mathcal{R}(A)}$ is a closed ideal of S , by our lemma 3.3, there exists an idempotent $e^2 = e \in \overline{\mathcal{R}(A)} - \mathcal{R}(A)$. Since $\mathcal{R}(A)$ is open, we have $B(\mathcal{R}(A)) = \overline{\mathcal{R}(A)} \cap \overline{S-\mathcal{R}(A)} = \overline{\mathcal{R}(A)} \cap (S-\mathcal{R}(A))$.

Consequently, we obtain that $e \in B(R(A))$. Now, let the maximal subgroup of S containing e the idempotent e by $H(e)$. $H(e)$ is compact because S is compact. As both $R(A)$ and $\overline{R(A)}$ are ideals of S , we thus obtain that $H(e) \cap R(A) = \phi$ and $H(e) \subset \overline{R(A)}$, which implies $H(e) \subset B(R(A))$, completing the proof.

Corollary 1 : If S is a bing (compact connected mob) and if $R(A)$ is a component of 0 , then $\overline{R(A)} - R(A)$ is a non-zero compact group.

This is a consequence of our theorem and a theorem of A.D. Wallace, [27], page 537.

Corollary 2 : Let the clan S (compact connected mob with unit) be contained in the Euclidean space E^n , $n \geq 2$. If A is an ideal of S such that $\overline{R(A)}$ includes the boundary of S , then $R(A)$ is a dense connected ideal of S .

Proof : Let $B(S)$ be the boundary of S in E^n . By our hypothesis, $B(S) \subset \overline{R(A)}$. This implies that $B(S) \subset SB(S) \subset S\,\overline{R(A)} \subseteq R(A)$, since S is a clan and $\overline{R(A)}$ is an ideal of S . Then $i^*: H^{n-1}(S) \cong H^{n-1}(\overline{R(A)})$ since, by a famous result of A.D. Wallace [28], $H^{n-1}(S,\overline{R(A)}) = 0 = H^n(S,\overline{R(A)})$. Hence by a result of [28] again, $\overline{R(A)} = S$, which means that $R(A)$ is dense in S . As S is connected, $R(A)$ is connected.

Remark : The stipulation "A is an ideal" of S cannot be weakened. The following example shows that even if A is an open subset of S but is not an ideal, our theorem 4.1 fails to be true.

Example 4.2. Let S be a comb space, that is,

$S = [0,\frac{1}{n}] \times [0,1] \cup [0,\frac{1}{2}] \times \{0\}$, for all $n=1,2,\ldots$. (see Figure III.)

where the underlined brackets denote the intervals, and define multiplication in S by:

$$<x_1,y_1> * <x_2,y_2> = <x_1x_2, \ \min\{y_1,y_2\}> \ .$$

S is an abelian mob.

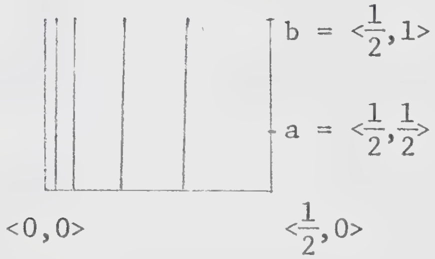


Figure III.

Now, let $A = \underline{(a,b)}$ where $a = <\frac{1}{2},\frac{1}{2}>$, $b = <\frac{1}{2},1>$, then $\mathcal{R}(A) = \underline{(a,b)}$, $B(\mathcal{R}(A)) = \{a\}$ which is not an idempotent of S .

If the compact group G does not lie in the boundary of $\mathcal{R}(A)$, then the compact group G must be contained in the ideal A . In fact, we have:

Theorem 4.3. Let S be a mob (not necessarily compact or abelian) and G be a compact group in S . Then G is entirely contained in the algebraic radical of an open ideal A of S if $G \cap A \neq \phi$.

Conversely, if $G \subset R(A)$, then $G \subset A$. (I am indebted to Professor J.M. Day for the improvement of this statement)

<u>Proof</u> : Suppose $G \cap A = \phi$, then $G \cap A$ is an open submob of G since A is open in S . By F.B. Wright [32], page 310, any open submob of a compact group is a closed subgroup, then $G \cap A$ is a closed neighborhood of the identity of the compact group G . Recall a famous result in topological groups, see [10], Theorem 7.6, page 61, we know that $G \cap A$ contains a normal subgroup N such that the quotient group $\frac{G}{N}$ is finite. Let k be the order of the group $\frac{G}{N}$. Then for any $g \in G$, we have $g^k \in N \subset G \cap A \subset A$. This means that $g \in R(A)$. In other words, $G \subset R(A)$. Conversely, let us suppose $G \subset R(A)$, then for every $g \in G$, $g \in R(A)$. Hence there exists an integer $k \geq 1$ such that $g^k \in A$. Since G is a group, we have $g^k = g' \in G$. Thus $G \cap A \neq \phi$. But A is assumed to be an ideal of S , so $G \cap A$ is an ideal of G and hence must equal to G if non-empty. This implies that $G \subset A$. Our proof is complete.

<u>Corollary</u> : In the above theorem, if A is an submob of S , then $G \subset R(A)$ if and only if $G \cap A \neq \phi$.

The following example shows that the converse part of theorem 4.3 is not true if A is not an ideal of S .

<u>Example 4.4.</u> Let S be the real interval $[-1,1]$. S with the usual topology and usual multiplication is clearly a mob.

Let $A = (-1,1]$, then $(A) = [-1,1]$ and $G = \{-1,1\}$ is a compact group in S . One can observe that $G \subset R(A)$ but $G \not\subset A$.

§5. Concluding remarks.

The definitions of reducibility and irreducibility of ideals can be generalized as follows: An ideal A is said to be R-irreducible if A is reducible such that $R(A_\alpha) = R(A)$. If $R(A_\alpha) \neq R(A)$ for all α , then A is said to be R-reducible. The following example shows that R-reducible ideals exist.

Example 5.1. Let S be the semigroup consisting of four elements $\{0,a,b,c\}$ such that $a^2 = a$, $c^2 = c$ and all other products are zero. Clearly, $\{0\}$, $\{0,a\}$, $\{0,c\}$ are ideals of S with $\{0\} = \{0,a\} \cap \{0,c\}$. But $R(\{0\}) = \{0,b\}$, $R(\{0,a\}) = \{0,a,b\}$, $R(\{0,c\}) = \{0,c,b\}$. Thus $\{0\}$ is R-reducible.

The following facts are easily verified.

(1) Any algebraic radical of an ideal which is open and non-prime in the compact mob S is R-reducible.

(2) If A is strongly reducible such that $R(A)$ is a maximal proper ideal of S , then A is R-irreducible.

(3) If a primary ideal is strongly reducible, then it is R-irreducible.

It should be pointed out that, in general, a primary ideal Q and its associated prime ideal $\mathcal{R}(Q)$ are topologically unrelated. For instance, the statement:

"Q compact if and only if $\mathcal{R}(Q)$ is compact" is not true. For Q is compact does not imply $\mathcal{R}(Q)$ is compact: Compare remark 1 in §3 . Also $\mathcal{R}(Q)$ is compact does not imply Q is compact. For take $S = [0,1]$ with the multiplication $*$ defined by $x * y = \frac{1}{2}xy$ for all x,y in S . Then $Q = [0,\frac{1}{2})$ is a primary ideal of S which is not compact while $\mathcal{R}(Q) = [0,1]$.

Also "Q is connected if and only if $\mathcal{R}(Q)$ is connected" is not true. For take $S = [0,\frac{1}{2}] \cup [1,2]$. Define $x * y = \frac{1}{8}xy$ for all x,y in S . Then $[0,\frac{1}{2}]$ is a primary ideal of S , $\mathcal{R}([0,\frac{1}{2}]) = S$ is disconnected. On the other hand, take $S = \{(x,y) \mid 0 \leq x \leq 1 , 0 \leq y \leq 1\}$. Define $(x,y) * (x',y') = (0,yy')$. Let $Q = \{(x,y) \mid x \in \ 0,1\} , 0 \leq y \leq 1\}$. Then it can easily be checked that Q is a primary ideal of S . As $\mathcal{R}(Q) = S$, $\mathcal{R}(Q)$ is connected, however, Q itself is disconnected. But if S is connected and Q is a connected ideal of S , then $\mathcal{R}(Q)$ is a connected ideal, since S connected and has zero, every ideal of S is connected.

CHAPTER II

On topological radicals in mobs

In the previous chapter, we investigated the structure of algebraic radicals in mobs and extended some results in commutative algebras to compact abelian mobs with zero. In this chapter, we study the topological radical of a mob. Several results converning abelian mobs with zero and local zeros are obtained.

By the topological radical of a mob S with zero, we mean the union of all the nil ideals of S. An element b in S is called nilpotent if $b^n \to 0$, that is, if for every neighborhood U of 0, there exists an integer n_o such that $b^n \in U$ for every $n \geq n_o$. We denote by N the set of all nilpotent elements of S. An ideal A of S consisting entirely of nilpotent elements is called a nil ideal of S. In case N is an ideal of S, then N turns out to be our topological radical of S. The concept of nil ideals was first introduced by K. Numakura in 1951 [18]. In his paper, he investigated the structure of S when N is open. Some amplification of his results on compact mobs with zero were given by R.J. Koch [13].

It is the purpose of this chapter to apply the work of Numakura [18] on mobs with zero to compact abelian mobs with zero and local zeros.

We are mainly interested in studying some properties of the set N. We prove that if N can be embedded densely in an abelian mob S, then S has no local zeros. We also show that if a mob S contains zero and local zeros, then S must be disconnected. Some conditions which lead to the existence of a local zero in a compact N-mob are given. A characterization of compact abelian N-mobs is obtained. Moreover, if S is a compact Ω-mob with zero such that $\overline{N} = N$, then $\overline{N} - N$ is either a group or a semilattice of groups. The set of topological zero divisors of an element a in S will also be treated.

Since life is considerably different if a mob is not compact than when it is, we therefore make it clear when we are assuming compactness on a mob, and when we are not. Throughout, for sets $X, Y \subset S$, $X-Y$ denotes the complement Y in X, XY denotes the set of all products xy with $x \in X$, $y \in Y$, $\overline{X}$ denotes the closure of the set X in S. All spaces are topological Hausdorff in this chapter. Unless otherwise stated, S will always be regarded as a topological abelian mob with zero. The reader is referred to [21] for terminology and notations.

§1. Definitions and Preliminary Results.

In this section, N denotes the set of all nilpotent elements in S and S denotes a non-empty abelian mob with zero. To avoid trivialities, we suppose that $S \neq \{0\}$ and the space S has at least three points.

<u>Notation.</u> Let A be a subset of S .

$J(A) = A \cup AS$, that is, the smallest ideal containing A .

$J_o(A)$ = the union of all ideals contained in A , that is, the largest ideal contained in A if there are any.

<u>Definition 1.1.</u> S is said to be an N-mob if N is an open subset of S .

S is said to be an Ω-mob if for any two ideals I_1 and I_2 such that $I_1 \cap I_2 \neq \phi$, then either $I_1 \subset I_2$ or $I_2 \subset I_1$.

<u>Definition 1.2.</u> An element 0 such that $a0 = 0a = 0$ for all a in S is called a zero element of S and it is easily seen that 0 is uniquely defined if it exists.

An idempotent element $e \neq 0$ is called a local zero if there exists an open neighborhood U in S such that $e \in U$ and e is the zero for U , that is, $ex = xe = e$ for every $x \in U$. We observe that a zero is not a local zero.

<u>Definition 1.3.</u> Let a be an element of S . Define Tod a = $\{x \in S \mid ax \in N\}$, that is, the set of all topological zero divisors of S . Tod a is non-empty since $\{0\}$ is always in Tod a . Let A be a subset of S , then Tod $A = \bigcup_{a \in A}$ Tod a .

<u>Definition 1.4.</u> Let $k \geq 1$ be an integer. A k-ideal A of S is a non-vacuous subset of S such that $A^k S \subset A$. A principal k-ideal generated by any subset A of S is the set $J_k(A) = A \cup A^2 \cup \ldots \cup A^k S$, which is the smallest k-ideal containing the sets $\{A, A, \ldots, A^k\}$. We note that $J_k(A)$ is not the set $A \cup A^k S$ unless A is a submob of S.

The following results are elementary, but they are useful. Some important properties of the set N are known after these propositions and counter-examples.

<u>Proposition 1.5.</u> Let S be a mob (not necessarily compact), then the following are true.

(i) The set N is always a submob of S. Moreover, N contains $\Gamma(a)$ for each $a \in N$.

(ii) Let S have a unit u. Then $u \in \text{Tod } a$ if and only if $a \in N$, for any $a \in S$.

(iii) If e is an idempotent element of S, then $N \subseteq \text{Tod } e$.

(iv) If N is an ideal of S, then $\text{Tod } e$ is an ideal of S. Moreover, $\text{Tod } n = S$ for all $n \in N$.

(v) If each $\text{Tod } a$ is an ideal of S for every non-zero $a \in S$, then every principal k-ideal $(k > 1)$ generated by an element $n \in N$ is contained in N.

(vi) If N is connected and S contains at least an idempotent $e \neq 0$, then every element of N is contained in a connected ideal of N.

We only prove (v) - (vi) ; the others are direct consequences of our definitions and we leave them to the reader.

Proof : (v) Take $n \in N$, and let $k > 1$ be an integer. Then $n^k \in N$. Hence $n \in \text{Tod } n^{k-1}$. By our assumption, Tod n^{k-1} is an ideal of S . Thus for any $y \in S$, $ny \in \text{Tod } n^{k-1}$. That is, $n^k y \in N$. In otherwords, $n^k S \subset N$. Since the principal k-ideal generated by the element n is the set $J_k(n) = n \cup n^2 \cup \ldots \cup n^k \cup n^k S$, clearly, $J_k(n) \subset N$.

Remark : I cannot prove the case when $k = 1$. If $k = 1$ is true, then N is an ideal of S .

(vi) To prove (vi), we first note that Tod e is a submob of S . For let $x \in \text{Tod } e$ and $y \in \text{Tod } e$. Then we have $ex \in N$ and $ey \in N$. Since S is abelian, by (i), N is a submob of S . Thus $exy = e^2 xy = (ex)(ey) \in N^2 \subset N$. Hence $xy \in \text{Tod } e$. Now let us denote Tod e by T . By (iii), we have $N \subset T$ and $0 \in T$. Let $\mathcal{D}$ be the component of zero in T . Then $0 \in \mathcal{D} \subset T$ which is a maximal connected set contained in T . Since N is connected, we have $\{0\} \in N \subset \mathcal{D}$. For any element $y \in N$, $\mathcal{D}y$, being the continuous image of a connected set, is connected and contains zero. By the maximality of the set $\mathcal{D}$, we have $\mathcal{D}y \subset \mathcal{D}$. So $\mathcal{D}$ is a connected ideal of N and every element of N is contained in $\mathcal{D}$. Our proof is completed.

Let $\Gamma(x) = \{x^n\}_{n=1}^{\infty}$, if $\Gamma(x)$ is compact for each element x is S , then S is called elementwise compact.

<u>Theorem 1.6</u>. If S is an elementwise compact (or sequentially compact) abelian mob, then N is an ideal of S .

<u>Proof</u> : (i) Suppose S is elementwise compact. We wish to show that N is an ideal of S . Let $x \in N$ and $y \in S$. Consider $\Gamma(y)$, which by our assumption, is compact. Take and $z \in \Gamma(y)$. We have $z0 = 0$. Thus, by the continuity of multiplication, for any arbitrary neighborhood U of 0 , there exist neighborhood $w(z) \in G(z)$, $w_Z(0) \in G(0)$ such that $w(z)\, w_Z(0) \subset U$, where $G(Z)$, $G(0)$ are complete systems of neighborhoods of the elements z and 0 respectively. Let us consider a system of neighborhoods $\{w(Z) | Z \in \Gamma(y)\}$. It is evident the $\Gamma(y) \subset \bigcup_{Z \in \Gamma(y)} w(Z)$. Since $\Gamma(y)$ is a compact mob, there exists a finite system $w(Z_1), w(Z_2),\dots,w(Z_n)$ which also covers $\Gamma(y)$ and for $i = 1,2,\dots,n$, we have $w(Z_i)\, w_{Z_i}(0) \subset U$. Evidently, there exists a neighborhood $w(0) \in G(0)$ such that $w(0) \subset \bigcup_{i=1}^{n} w(Z_i)$ and $w(Z_i)\, w(0) \subset U$ for every $i = 1,2,\dots,n$. But $w(0)$ is a neighborhood of 0 and $x \in N$, and hence $x^n \in w(0)$ for $n \geq n_o$ for some n_o . Thus $y^n x^n \in \Gamma(y)\, w(0) \subseteq \cup\{w(Z_i) | i=1,2,\dots n\}\, w(0) \subset \cup\, U = U$, for $n \geq n_o$. This means that $xy \in N$, that is, N is an ideal of S .

(ii) Now suppose S is sequentially compact and N is not an ideal of S .

Then we can find $x \in N$ and $y \in S$, such that $xy \notin N$. That is, $(xy)^n \not\to 0$. Thus for any open neighborhood V of $\{0\}$, there exists a subsequence $(xy)^{n_k}$ of $(xy)^n$ such that $(xy)^{n_k} \notin V$ for every $k=1,2,\ldots(*)$. Consider the subsequence $\{y^{n_k} | k=1,2,\ldots\}$. Since S is sequentially compact and Hausdorff, there exists a subsubsequence $\{y^{n_{k_i}} | i=1,2,\ldots\}$ of $\{y^{n_k}\}$ such that $y^{n_{k_i}} \to s \in S$. Clearly, $\lim_i (xy)^{n_{k_i}} = \lim_i (x^{n_{k_i}} y^{n_{k_i}}) = \lim_i x^{n_{k_i}} \lim_i y^{n_{k_i}} \to 0.S = 0$. This, however, contradicts to $(*)$. So, indeed, N is an ideal of S.

<u>Corollary 1.</u> If S is a compact abelian mob with zero, then N is an ideal of S.

<u>Corollary 2.</u> If S is compact and connected and if $S^2 = S$, then N is connected.

<u>Proof</u> : By Koch and Wallace [14], page 682, S compact with $S^2=S$ implies $ES = S$. Thus S has a unit since S is abelian. As S is also connected, so by Koch and Wallace [14], page 683, each ideal of S is connected. By Corollary 1, N is an ideal of S, so N is therefore connected.

1.7 <u>Counter-examples.</u>

<u>Example (1).</u> If S is not compact, then N is not necessarily an ideal of S.

For example, let $S = [0,\infty)$ with usual topology and usual multiplication. Then $N = [0,1)$, which is not an ideal of S.

Example (2). If S is not an abelian mob, then N is not necessarily a submob of S. For example, let $S = \{0 = \begin{pmatrix}00\\00\end{pmatrix}$, $x = \begin{pmatrix}01\\00\end{pmatrix}$, $y = \begin{pmatrix}00\\10\end{pmatrix}$, $c = \begin{pmatrix}10\\00\end{pmatrix}$, $d = \begin{pmatrix}10\\01\end{pmatrix}\}$. Then under the ordinary matrix multiplication, we have the following multiplication table.

.	0	x	y	c	d
0	0	0	0	0	0
x	0	0	c	0	x
y	0	d	0	y	0
c	0	x	0	c	0
d	0	0	y	0	d

Clearly, $N = \{0,x,y\}$, $N^2 = \{0,c,d\}$. Thus $N^2 \nsubseteq N$. This example demonstrates that the condition abelian is necessary.

Example (3). Even if S is an abelian mob, N is not necessarily an idempotent set. For let S be the real line with the usual topology. Define $x*y = 0$ for all x,y in S. Then $N = S$ but $N^2 = 0$.

Example (4). Even if S is connected, Tod e is not necessarily connected. For let $S = \{-1\} \cup [-\frac{1}{2}, \frac{1}{2}] \cup \{1\}$. Define $x*y = xy$ for x and $y \geq 0$.

$$x*y = -xy \qquad \text{for } x \text{ and } y \leq 0$$

$$x*y = 0 \quad \text{for } x > 0,\ y \leq 0 \text{ or } y \geq 0,\ x < 0$$

then Tod $\{1\} = \{-1\} \cup [-\frac{1}{2},\frac{1}{2}]$, which is not connected.

<u>Example (5).</u> Even if S is compact connected, but $S^2 \neq S$ then N is not necessarily connected. For let S be the two line segments joining the points $(0,0)$, $(1,0)$ and $(1,0)$, $(0,1)$. The topology is the usual topology inherited from the planes. Define the multiplication on S by $(x_1,y_1) * (x_2,y_2) = (\min x_1 x_2, 0)$. Then, S is clearly a connected abelian mob with zero. But $N = \{(0,0), (0,1)\}$ which is disconnected.

<u>Example (6).</u> k-ideals are natural extension of the ordinary ideals. We give some examples:

<u>Example (6.1).</u> Let $S = \{0,1,2,3,4,5\}$ be a semigroup with the following multiplication table.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	4	0	2
2	0	0	0	0	0	0
3	0	5	0	0	2	0
4	0	2	0	0	0	0
5	0	0	0	2	0	0

Let $A = \{0,1,2,3,4\}$, then $A^2 = \{0,2,4,5\}$. Clearly, $AS \not\subset A$, but $A^2S = \{0,2,4\} \subset A$.

<u>Example (6.2).</u> Let $S = \{3 \times 3$ matrices$\}$. Then S is a mob with ordinary matrix multiplication. Let $A = \{\theta, a, a^2\}$, where θ is the 0-element and $a = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then A is a submob of S. It is clear that $A^2S \not\subset A$, but $A^3S = \{\theta\} \subset A$.

Example (6.3). Let $S = [0,2)$ with usual topology. Define the multiplication on S by $x*y = xy \pmod 2$. Then S is a mob. Let $A = [0, \frac{3}{4})$, then $A^3S \not\subset A$, but $A^4S \subset A$.

Example 7. We give here a simple example to support the statements of our proposition 1.5 (iii)-(v) .

Let $S = \{0,a,e,f\}$ be a semigroup with the multiplication table

.	0	a	e	f
0	0	0	0	0
a	0	0	0	a
e	0	0	e	0
f	0	a	0	f

The lattice of ideals of S is given by the graph in Figure I.

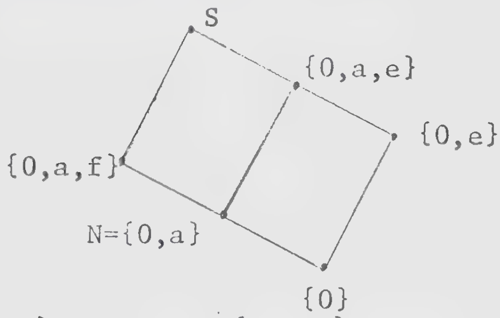


Figure I.

Clearly, $N = \{0,a\}$, Tod $e = \{0,a,f\}$, Tod $f = \{0,a,e\}$ = Tod N .

The largest ideal which has a zero intersection with N is $\{0,e\}$. Also we note that S contains two minimal ideals, namely, $\{0,e\}$ and $\{0,a\}$ both of them are contained in Tod $N = \{0,a,e\}$. Moreover, $\{0,e\}$ is a minimal non-nil ideal of S .

If N is the topological radical of S and G is any non-zero subgroup of S, then $G \subset N'$. If $N' = \{e\}$, $e^2 = e \neq 0$, then $N = \text{Tod } e$.

Proof : To prove that N is the topological radical of S, it suffices to show that N is an ideal of S. But this is trivial since N' consists only of idempotent elements. If N is the topological radical of S, then N, in particular, is an ideal of S. Now let us suppose that $N \cap G \neq \phi$. Then there exists $x \in G \cap N$. As G is a subgroup of S, $Gx = xG = G$. Hence $G = Gx \subset SN \subset N$ which implies $G = \{0\}$. Our supposition is therefore impossible. As $N \subset \text{Tod } e$ by 1.5 (iii), so to prove $N = \text{Tod } e$, it suffices to show that $\text{Tod } e \subset N$. Suppose that $y \notin N$, then $y \in N' = \{e\}$. This means that $y = e \notin \text{Tod } e$. Our assertion is proved.

Proposition 1.9. Let a be any arbitrary element in S. If N is an open subset of S, then Tod a is open. If N is a closed subset of S, then Tod a is closed.

Proof : For each $x \in \text{Tod } a$, we have $ax \in N$. According to our assumption, N is open and hence there exists a neighborhood $V(ax)$ of ax such that $V(ax) \subset N$.

From the continuity of multiplication, there exists a neighborhood $V_1(a)$, $V_2(x)$ of the points a, x respectively such that $V_1(a)\, V_2(x) \subset V(ax) \subset N$. Hence $aV_2(x) \subset V_1(a)\, V_2(x) \subset N$. This means that $x \in V_2(x) \subset$ Tod a. Hence Tod a is an open subset of S. To prove that Tod a is closed if N is closed, it suffices to prove that $\overline{\text{Tod } a} \subset$ Tod a. Let $x \in \overline{\text{Tod } a}$. Then since S is Hausdorff, there exists a sequence $\{x_i\} \in$ Tod a such that $x_i \to x$. This implies that $ax_i \in N$ for each i and by the continuity of multiplication, we have $ax_i \to ax$. Since N is closed, we have $ax \in N$. This means that $x \in$ Tod a.

(I was told that Professor A.D. Wallace also obtained the same results in [2] and [31]. He denotes Tod a by $a^{[-1]}N$.)

<u>Remark 1.10.</u> The following example shows that if N is not open, then not all Tod a are open sets.

Let $S = \{Z^-\} \cup [0,\infty)$, where $\{Z^-\}$ are the negative integers. The topology of S is the usual topology inherited from the real line. Define the multiplication $*$ in S by

$x*y = \min\{x,y\}$ whenever $x,y \in [0,\infty)$.

$x*y = 0$ if $x \in \{Z^-\}$, $y \in [0,\infty)$ and vice versa.

$x*y = -xy$ if both $x,y \in \{Z^-\}$ where xy is the usual multiplication.

Clearly $N = \{0\}$ is not open. For any $a \in [0,\infty)$, Tod $a = \{0\} \cup \{Z^-\}$, which is open subset of S. For any $a \in \{Z^-\}$, Tod $a = [0,\infty)$ which is not open.

Now let us call a mob S an A-mob if all Tod a are open subsets of S for every $a \in S$. From proposition 1.9, we know that if S is an N-mob, then S is an A-mob. But the converse statement is not known to the author. That is, if S is an A-mob, is S an N-mob? However, Dr. C.S. Hoo gives a partial answer to this question.

<u>Theorem 1.10.</u> If S is a compact A-mob and E is finite, then S is an N-mob.

<u>Proof</u> : It is true that $N \subset \bigcap_{e \in E}$ Tod e. We only need to show that $\bigcap_{e \in E}$ Tod $e \subset N$. Let $x \in \bigcap_{e \in E}$ Tod e, then $x \in$ Tod e for all $e \in E$. So $ex \in N$ for all $e \in E$. Now there exists $e_1 \in E$ such that $e_1 \in \Gamma(x)$ [21], page 22. As $K(x) = e\,\Gamma(x)$ and is a group [21], page 24, so $e_1 x \in K(x)$ and $e_1 x$ has an inverse element $y \in K(x)$. By our theorem 1.6, N is an ideal of S. Thus, we have $e_1 = e_1 xy \in NS \subset N$. This implies that $e_1 = 0$, that is, $K(x) = \{0\}$. But $K(x)$ is the set of cluster points of the sequence $\{x^n\}_{x=1}^{\infty}$. Hence $x^n \to 0$, that is, $x \in N$. Therefore $N = \bigcap_{e \in E}$ Tod e. If E is finite, then we know immediately that A-mob implies N-mob.

<u>Corollary 1.</u> If S is a compact mob and Tod e is closed for all $e \in E$, then N is closed.

This corollary gives a converse to our proposition 1.9.

Corollary 2. $N = \bigcap_{e \in E^+} \text{Tod } e$, where E^+ is the set of all non-minimal idempotents of S .

Proof : By our theorem, we know $N = \bigcap_{e \in E^*} \text{Tod } e$, where $E^* = E - \{0\}$ since $\text{Tod } 0 = S$. Now suppose e, e' are idempotents of S . Let us suppose $e \leq e'$, that is, $ee' = e$. Then if $x \in \text{Tod } e'$, we have $e'x \in N$. Thus $(ee')x = e(e'x) \in eN \subset N$. That is, $ex \in N$. Hence $x \in \text{Tod } e$. We therefore obtain that if $e \leq e'$, then $\text{Tod } e' \subset \text{Tod } e$. Thus, $N = \bigcap_{e \in E^+} \text{Tod } e$.

Corollary 3. Let $e \in E$ and $Pe = \{x \in S | e \in \Gamma(x)\}$. If $e \neq 0$, then $Pe \in \text{Tod } e = \phi$.

The following gives a necessary and sufficient condition for the set N to be a k-ideal of S .

Theorem 1.11. Let S be connected and let $J_k(N)$ be the k-ideal generated by N . Then N is a k-ideal of S if and only if the component $\mathcal{C}$ of $\{0\}$ in N coincides with the the component $\mathcal{D}$ of $\{0\}$ in $J_k(N)$.

To prove this theorem, we prove something more general, namely, if we replace the set N by any submob A of S containing zero, we shall see that our statement still holds.

Proof : Since A is a submob of S, we have $J_k(A) = A \cup A^2 \cup \ldots \cup A^kS = A \cup A^kS$. Suppose A is a k-ideal of S. Then $A^kS \subset A$ and we have $J_k(A) = A$. Thus, the component $\mathcal{C}$ of $\{0\}$ in A and the component $\mathcal{D}$ of $\{0\}$ in $J_k(A)$ coincides. For the converse part, we first observe that $0 \in AS$. Also, for any $a \in A$, the point $(a,0) \in (\{a\} \times S) \cap (S \times \{0\})$. Since S is connected, $(\bigcup_{a \in A} (\{a\} \times S) \cup (S \times \{0\}) = (A \times S) \cup (S \times \{0\})$ is a connected subset of $S \times S$. By the continuity of multiplication, we have that AS is a connected subset of S containing 0. As a consequence, $0 \in A^kS$ and A^kS is connected. Moreover, $J_k(A) \supset A^kS$. Thus the component $\mathcal{D}$ of $\{0\}$ in $J_k(A)$ contains A^kS. On the other hand, $A \subset J_k(A)$ and $\{0\} \in A$. Thus the component $\mathcal{C}$ of A is also in $J_k(A)$. By our assumption, $\mathcal{D} = \mathcal{C}$. Thus $A^kS \subset \mathcal{D} = \mathcal{C} \subset A$, that is, A is a k-ideal of S.

§2. Abelian N-mobs.

We now study, in this section, the structure of N-mobs, that is, topological semigroups S in which the set N is an open subset of S. We use E to represent the collection of all idempotents of S and $E^* = E - \{0\}$. It is easily seen that E is closed [18], page 408. An idempotent e may be looked upon as a subgroup of S. Following the usage of [11] and elsewhere, $H(e)$ is the maximal group containing an idempotent e.

Also, following the notation of R.J. Koch [13] and K. Numakura [18], we define, for each x in S, $\Gamma_n(x) = \{x^i\}_{i=n}^{\infty}$, $\Gamma(x) = \Gamma_1(x)$, $K(x) = \cap \{\Gamma_n(x) \mid n \geq 1\}$ and in case $\Gamma(x)$ is compact, we have $\Gamma(x) = \{x, x^2, \ldots\} \cup K(x)$ where $K(x)$ is the minimal ideal of $\Gamma(x)$ and is the maximal subgroup of $\Gamma(x)$.

<u>Theorem 2.1.</u> If S is a compact N-mob (not necessarily abelian) which is not nil, then there always exists a compact subgroup of S which is disjoint from N.

<u>Proof</u> : Since S is not nil, there exists at least an element x S with $x \in N$. By lemma 2.1.4 of [21], page 58, $x \notin N$ implies $x^n \notin N$ for all positive integers n. Thus the sequence $\{x^n\}_{n=1}^{\infty} \subset N'$. Since N is open, N' is closed and hence compact. Therefore, $\Gamma(x) \subset N'$. Clearly, $\Gamma(x)$ is compact. So there exists a unique idempoent $0 \neq e^2 = e \in \Gamma(x)$. Consider $H(e)$ in $\Gamma(x)$. Since S is compact, $H(e)$ is a non-zero compact group of S. It is clear that $H(e) \cap N = \phi$, for $H(e) \subset \Gamma(x) \subset N'$.

One would naturally ask: Under what conditions, can S be uniquely decomposed into two disjoint sets N and G, where N is the set of nilpotent elements and G is a compact subgroup of S, that is, we need $S = N \cup G$, $N \cap G = \phi$. Algebraically, we can construct the following:

<u>Theorem 2.2.</u> Let N be an abelian semigroup, G an abelian group which is disjoint from N.

Define the multiplication $\odot$ in the set $S = G \cup N$ as follows:

(a) For $x, y \in G$, let $x \odot y = x * y$, where $*$ is the group multiplication.

(b) For $x, y \in N$, let $x \odot y = x.y$, where $.$ is the semigroup multiplication.

(c) For $x \in G$, $y \in N$, let $x \odot y = y = y \odot x$.

Then S is an abstract abelian semigroup, denoted by $S(N,G;\odot)$, in which N is the unique maximal proper ideal of S. In other words, N can be embedded as the unique maximal proper ideal in $S(N,G;\odot)$.

<u>Proof</u> : One can easily verify that the multiplication $\odot$ of S is associative, commutative and closed, hence $S(N,G;\odot)$ is an abstract abelian semigroup. Also, it is easily seen that N is an ideal of S. We only have to show that N is the unique maximal proper ideal of S. In fact, suppose A is an ideal such that $A \not\subset N$. Then there exists $a \in A$, $a \notin N$. Thus, for each $x \in N$, we have $a \odot x = x \in AN \subset A$, that is, $N \subset A$. Suppose $N \subsetneq A \subset S$. Then $G \cap A \neq \phi$. Let $b \in G \cap A$. As G is a group, $Gb = G \subset SA \subset A$. Hence $S = G \cup A \subset A$ which implies $S = A$. Thus $A \not\subset N \Rightarrow A = S$, and N is indeed the unique maximal ideal of S.

We are now able to give an answer to our question raised above.

<u>Theorem 2.3.</u> (J.M. Day). Let S be a compact mob with only one non-zero idempotent,

then S is the disjoint union of N and a compact group G if and only if N is the maximal proper ideal of S.

Proof : Apply our theorems 2.1 and 2.2. The only thing we need to observe is that: If N is a maximal proper ideal of S, then N is open.

We give here a characterization of abelian N-mobs.

Theorem 2.4. Let S be an abelian compact mob which is not nil. Then S is an N-mob if and only if E^* is compact.

In order to prove this theorem, our theorem 1.6 is crucial.

Proof : Let us suppose that S is an abelian compact N-mob. Then N is an open subset of S. Clearly, $E^* = N' \cap E$. As N is open, N' is closed. Also, it is well known that E is closed [18], page 408, so by the compactness of S, N' and E are compact subsets of S. Therfore E^* is compact. Conversely, let us suppose E^* is compact. Then $S-E^*$ is open and $N \subset S-E^*$. Clearly, $J_o(S-E^*)$, the union of all ideals of S contained in $S-E^*$, is open [14], page 681. By our Theorem 1.6, N is an ideal of S and hence $N \subset J_o(S-E^*)$. Suppose if possible, $N = J_o(S-E^*)$. Then there exists $x \in J_o(S-E^*)$, $x \notin N$. Consider the principal ideal $J(x)$ generated by x. Then $J(x) = x \cup xS \subset J_o(S-E^*)$. Since S is compact, so is $J(x)$.

Hence $\Gamma(x) \subset J(x)$ and there exists a unique idempotent element $e^2 = e \in \Gamma(x) \subset J_o(S-E^*)$ [21], page 22. Clearly $e = 0$ for otherwise $e \not\in J_o(S-E^*)$. But if $e = 0$, then $K(x) = e\,\Gamma(x) = \{0\}$. Since $K(x)$ is the set of cluster points of the sequence $\{x^n\}_{n=1}^{\infty}$, we would have $x^n \to 0$. This contradicts $x \not\in N$. Thus we conclude that $N = J_o(S-E^*)$, which is an open subset of S .

Corollary 1. Let S be compact. If there exists an idempotent $e \neq 0$ such that $N \neq \text{Tod } e$, then $\text{Tod } e$ contains at least one non-zero idempotent of S .

Proof : By Theorem 1.6, N is an ideal of S . Since $N \neq \text{Tod } e$ so by proposition 1.5 (iii) and (iv) , $N \underset{\neq}{\subset} \text{Tod } e$ and $\text{Tod } e$ is and ideal of S . Applying the same argument as in the proof of Theorem 2.3, we see that there exists $0 \neq f^2 = f \in \Gamma(x) \subset \text{Tod } e$.

Corollary 2. If S is compact and $\mathcal{D}$ is the component of 0 in N , then $NS \subset \mathcal{D}$.

Corollary 3. Let S be compact. If $E \subset \overline{N}$ and $\overline{N} \neq S$, then $\overline{N}$ is contained in a proper compact ideal of S .

Proof : Suppose $\overline{N} \neq S$. According to our lemma 2.4, $\overline{N}$ is a proper ideal of S . Thus $\overline{N}$ is contained in all proper maximal ideals M_i of S , where M_i has the form $J_o(S-x_i)$ for some $x_i \in S-\overline{N}$. Hence $E \subset \overline{N} \subset M_i$ for all i . By Koch and Wallace [14], page 683, we have $S^2 \subset M_i$ for all i , that is $S^2 \subset \bigcap_i M_i$.

But since S is compact, by Koch and Wallace again [14], page 638, $\bigcap_i M_i \subset S^2$. Hence $S^2 = \bigcap_i M_i$. Clearly, $\bigcap_i M_i$ is compact and $\overline{N} \subset \bigcap_i M_i$.

<u>Theorem 2.5.</u> If S is a compact Ω-mob such that $\overline{N} \neq N$, then $\overline{N} - N$ is either a group or a semilattice of groups.

To prove theorem 2.5, we need the following:

<u>Lemma 2.6.</u> Let S be a compact Ω-mob. If I is a minimal ideal containing N properly, then I is closed. Moreover, $I-N$ is a closed non-nil submob of S if and only if $I^2 \nsubseteq N$.

<u>Proof</u> : One part is trivial. For the converse part, let us consider $I-N$. We claim that $I-N$ is a submob of S . For let $a,b \in I-N$. Then $a^n, b^n \in I-N$ for every $n = 1,2,\ldots$. Now, let $ab \in N$. Consider $J(a) = a \cup Sa \subset I$, which is the principal ideal generated by a . As $a \nsubseteq N$ we have $J(a) \nsubseteq N$. Since S is an Ω-mob, we obtain $N \subsetneqq J(a) \subset I$. However, by the minimality of I containing N , we conclude that $J(a) = I$, hence I is closed. Similarly , $J(a) = J(a^n) = I$ for every $n = 1,2,\ldots$, $J(b) = J(b^n) = I$ for every $n = 1,2,\ldots$. Now $J(a)\,J(b) = (a \cup Sa)(b \cup Sb) = ab \cup Sab = J(ab)$. So $I^2 = J(ab) \subset I$. According to our assumption, $I^2 \nsubseteq N$. Therefore, we have $N \subsetneqq I^2 \subset I$, which implies $J(ab) = I$. Similarly, $J((ab)^n) = I$ for every $n = 1,2,\ldots$. Hence $\bigcap_n J((ab)^n) = I$.

Since $ab \in N$ $(ab)^n \to 0$, we obtain that $I = \bigcap_n J((ab)^n) = \{0\}$ which is a contradiction, for we assumed that I contains N properly. This establishes our claim. Now take $a \in I-N$. Then $a, a^2, \ldots$ are all in $I-N$. Hence $\Gamma(a) = \overline{\{a^n\}_{n=1}^{\infty}} \subset J(a)$ and $\Gamma(a)$ is compact. Thus there exists an unique idempotent $f = f^2 \in \Gamma(a) \subset I$. Since $a \notin N$, by the same argument as in our Theorem 2.4, we see that $f \neq 0$. Hence $I-N$ contains a non-zero idempotent and $I-N$ is a closed non-nil submob of S.

We now prove theorem 2.5. By Theorem 1.6, $\overline{N}$ is an ideal of S which contains N properly. We claim that $\overline{N}$ itself is the minimal ideal containing N properly. For suppose that there exists an ideal N_1 such that $N \subsetneqq N_1 \subset \overline{N}$. Take $x \in N_1 - N$. Then as S is an Ω-mob, we have $N \subsetneqq J(x) = x \cup xS \subset N_1 \subset \overline{N}$. As S is compact, $J(x)$ is closed. Hence by the definition of the closure of N, $\overline{N} = J(x)$, which implies $N_1 = \overline{N}$. Now applying our Theorem 1.6, $\overline{N}$ is a minimal non-nil ideal of S. So there exists a non-zero idempotent $e^2 = e \in \overline{N} - N$ such that $J(e) = e \cup Se = \overline{N}$. This implies that $\overline{N} = eS$. Hence, by a result of R.J. Koch [21], page 57, $eS - N$ is a group and e is primitive. Moreover, if $\overline{N}$ contains more than one idempotent, then $eS - N$ is the disjoint union of the maximal groups $e_\alpha S-N$ for all $e_\alpha \in \overline{N} - N$ [21], page 61. In other words, $eS-N$ is either a group or a semilattice of groups.

The following is a slight modification of a theorem given by K. Numakura [18], page 407, theorem 4.

However, for the sake of completeness, we give the proofs in detail.

Theorem 2.7. If S is locally compact and N is a compact ideal of S, then for any open neighborhood V of N, there always exists an open non-nil submob J of S such that $N \subset J \subset V$.

Proof : Since S is locally compact and Hausdorff, S is regular and we can find a neighborhood U of N having compact closure such that $N \subset U \subset \overline{U} \subset V$, where V is any open neighborhood containing N. Since N is an ideal, so $N \overline{U} \subset N \subset U$. By the continuity of the multiplication and compactness of $\overline{N}$ and $\overline{U}$, we can find an open set W with $N \subset W \subset U$, and $W \overline{U} \subset U$. Since $W \subset \overline{v}$, $W^2 \subset W\overline{U} \subset U$. Similarly, $W^3 \subset U, \ldots$ and hence $\bigcup_n W^n \subset U$. Denote $T = \overline{\bigcup_n W^n}$. T is clearly a compact submob of S contained in V. Now let $J = J_o(W)$, the union of all ideals contained in W. Therefore $J \subset W \subset T \subset V$. Since T is compact, J is therefore open and is a submob of S. Since N is an ideal contained in W, so $N \subsetneq J \subset W \subset V$. Clearly J is non-nil, for otherwise, we would have $J \subset N$, which is false.

§3. Abelian mobs with zero and local zeros.

In this section, we shall consider abelian mobs with zero and local zeros. Throughout, N will stand for the set of all nilpotent elements of the mob S.

<u>Theorem 3.1.</u> Let S be a mob (not necessarily compact). Then the closure of N contains no local zeros.

<u>Proof</u> : Suppose $\overline{N} = T$ and suppose T has a local zero e^*. As N is dense in T, then for any $t \in T$ and any neighborhood $V(t)$ of t, we have $V(t) \cap N \neq \phi$. In particular, as e^* is a local zero of T, there is a $V(e^*)$, $V(e^*) \cap N \neq \phi$ and that $x \in V(e^*)$ implies $xe^* = e^*x = e^*$. Let us that $x \in V(e^*) \cap N$, then $e^*x = xe^* = e^*$, and by definition of local zero, we have $e^* \notin N$. However, since $x \in N$, it is nilpotent. Thus $(e^*x)^k = e^*x^k \to 0$. This implies that $e^* = 0$, which contradicts to the definition of local zero. Therefore, we conclude that $\overline{N}$ has no local zeros.

The next theorem tells us that if S contains zero and local zeros, then S must be disconnected.

<u>Theorem 3.2.</u> If P is a component of a mob S (not necessarily compact) which has a local zero $e^* \in P$; then $Pe^* = e^*$ and $P \cap \overline{N} = \phi$.

<u>Proof</u> : We first show that e^* is the zero element for the subset P, that is, $Pe^* = e^*$. Now P is connected, hence e^*P is also connected, and is a subset of P. Clearly, $e^* \in e^*P$. As e^* is a local zero for S, then there is an open neighborhood U in S such that $e^* \in U \cap P = V$, $e^*x = xe^* = e^*$ for all $x \in V$. Hence $e^*P \cap V \neq \phi$. V is open in P.

Thus $e^*P \cap V$ is open in e^*P . Consider $x \in e^*P \cap V$ which implies that $x = e^*p$, $x \in V$ with $p \in P$, and this implies that $x = e^{*2}p = (e^*p)\,e^* = xe^* = e^*$. Hence, being in a Hausdorff space, $\{e^*\} = e^*P \cap V$ is both open and closed in e^*P , and hence equal to e^*P . That is, $e^*P = \{e^*\}$. Next, let us suppose that $P \cap \overline{N} \neq \phi$. Then there exists $x \in P \cap \overline{N}$. Hence $x \in P$ and $x \in \overline{N}$. Since e^* is the zero for P , so $xe^* = e^*$. Let U be a neighborhood of e^* so that e^* is the zero for U . By the continuity of the multiplication, there exists a neighborhood V of x such that $Ve^* \subset U$. But since $x \in \overline{N}$, so $V \cap N \neq \phi$. Let $y \in V \cap N$, then $y \in N$ and $ye^* \in U$. This implies that $(ye^*)e^* = e^*(ye^*) = e^*$. That is, $ye^* = e^*$. But then, $(e^*)^n = (ye^*)^n = y^ne^* \to 0$, which contradicts to the definition of local zero. So $P \cap \overline{N} = \phi$.

<u>Corollary 1.</u> If N' is a connected component of a compact mob S containing e^* , then

(i) N is closed

(ii) Tod e^* is a prime ideal containing N .

<u>Proof</u> : (i) Since S is compact, by Theorem 1.6, $\overline{N}$ is an ideal of S . Suppose on the contrary, $\overline{N} \neq N$. Then there exists an element $y \in \overline{N} - N$. So $e^*y \in \overline{N}$, since $\overline{N}$ is an ideal of S . By the theorem above, we have $e^*y = e^* \in \overline{N}$.

But our Theorem 3.1 tells us that $e^* \notin \overline{N}$, a contradiction, so N must be closed.

(ii) By proposition 1.5 (iii) and (iv), $N \subset \text{Tod } e^*$ and $\text{Tod } e^*$ is an ideal of S. We only need to verify that $\text{Tod } e^*$ is prime. For this purpose, let us consider $ab \in \text{Tod } e^*$, $a \notin \text{Tod } e^*$. Then it follows that $abe^* \in N$ and $ae^* \notin N$. Since $ae^* \notin N$ we have $ae^* \notin N'$. Now, by our theorem 3.2, e^* acts as a zero in N'. Hence $(ae^*)e^* = e^*$, that is, $ae^* = e^*$. Thus $abe^* \in N$ $b(ae^*) \in N$ $be^* \in N$ $b \in \text{Tod } e^*$. Thus $\text{Tod } e^*$ is indeed a prime ideal of S containing N.

<u>Corollary 2.</u> If e is any non-zero idempotent of a compact mob S such that $e^* \notin \text{Tod } e$, then $\text{Tod } e \cap C = \phi$, where C is the component of e^* in S.

The following theorem concerns the existence of a local zero in abelian compact mob with zero.

<u>Theorem 3.3.</u> Let S be a compact abelian N-mob. If $N' = E^*$, which is a connected subset of S, then S contains a local zero. Furthermore, if S has a unit and N' is arcwise, then N' is contratible.

The proof of this theorem is a consequence of the following.

<u>Theorem 3.4.</u> Let S be a compact connected mob (not necessarily abelian) such that $S = ES = SE$ and E is an abelian submob of S.

Then S has an idempotent e such that $eE = Ee = e$ and the minimal ideal $M(S) = H(e) = eSe = eS = Se$. Moreover $H^*(S)$ is isomorphic to $H^*(eSe)$.

(This theorem is due to Dr. C.S. Hoo.)

<u>Proof</u> : Let $K(S)$ be the minimal ideal of S . Then by 1.22 of [12], we can find a primitive idempotent e in $K(S)$ such that eSe is a group and $eSe = e\,K(S)e$. Let $H(e)$ be the maximal subgroup of S containing e . Then one can easily verify that $H(e) = eSe = eK(S)e$. By Theorem 1.2.11 of [21], page 34, we therefore have $K(S) = SeS$, which is a two-sided ideal of S . Hence by lemma 1.2.8 of [21] again, we have $K(S) = (Se \cap E)\, eSe\, (eS \cap E)$. We now show that under the condition of the theorem, $Se \cap E = eS \cap E = \{e\}$.

Suppose $e_1 \in Se \cap E$. Then we can write $e_1 = xe$ for some $x \in S$. Hence $ee_1 = exe \in eSe = H(e)$. Since E is abelian, ee_1 is an idempotent and hence $ee_1 = e$. Thus $e = xe = (xe)e = e_1e = e$. Similarly, we have $eS \cap E = \{e\}$. Thus we have established that $K(S) = eSe = H(e)$. Since $K(S)$ is closed, we have that $H^*(S) \simeq H^*(eSe)$. It remains for us to show that $eE = Ee = \{e\}$. Let $e_1 \in E$. Since $H(e) = K(S)$ is an ideal, we have $H(e)S \subset H(e)$, $SH(e) \subset H(e)$. Thus $ee_1 \in H(e)$, $e_1e \in H(e)$. Since ee_1 and e_1e are idempotents we have $ee_1 = e_1e = e$. It only remains to show that $K(S) = Se = eS$. In fact, let x be an element of $K(S)$. Then $x \in H(e)$, we have $x = ex \in Sx$. Thus $H(e) = K(S) \subset Sx$. On the other hand, $Sx \subset SK(S) \subset K(S)$. Thus $H(e) = K(S) = Sx$.

Similarly, $K(S) = xS$ for any $x \in K(S)$. In fact, for any $x \in K(S)$, it is easily seen that we have $xK(S) = K(S)x = K(S) = Sx = xS$.

We now prove theorem 3.3.

Since S is compact abelian N-mob, by theorem 2.4, we have $N' = E^*$ is compact and non-empty. Therefore E^* is a compact submob of S . The hypothesis of Theorem 3.4 on N' are satisfied. Hence we can find an idempotent e^* such that $e^*N' = N'e^* = \{e^*\}$, that is, S has a local zero e^* , and e^* is the zero of N' . Moreover, by Theorem 3.4, N' is acyclic. Now suppose S has a unit u . Then clearly, $u \in E^*$. As N' has a unit and is arcwise connected, we apply a result of Gottlieb and Rothman [8]. Recall that they say that the semigroup N' satisfies * if for each x in N' , there is an element y such that $xy = y$. Since N' has a zero, we see that N' satisfies * . Then by lemma 1 of [8], page 756, we have that N' is contractible.

<u>Theorem 3.5.</u> Let S (not necessarily compact) have a zero and local zero e^* . If $N' = E^*$ is connected, then $\operatorname{Tod} e^* = \bigcup_{a \in E^*} \operatorname{Tod} a$, in fact, $\operatorname{Tod} e^* = \operatorname{Tod} a$ for any $a \in E^*$.

<u>Proof</u> : Since $N' = E^*$, by proposition 1.8, N is an ideal of S.

By proposition 1.5 (iv), $\mathrm{Tod}\ x$ is an ideal of S for every $x \in S$. As $e^* \in E^*$ and E^* is connected, applying theorem 3.2, we can prove that $\mathrm{Tod}\ a \subset \mathrm{Tod}\ e^*$ for all $a \in N'$. Suppose if possible, $\mathrm{Tod}\ a \subsetneq \mathrm{Tod}\ e^*$ for all $a \in N'$, then there exists $x \in \mathrm{Tod}\ e^*$, $x \notin \mathrm{Tod}\ a$. As $\mathrm{Tod}\ e^*$ is an ideal of S, we have $xa \in \mathrm{Tod}\ e^*$, which implies $axe^* \in N$. But since $x \notin \mathrm{Tod}\ a$, we have $ax \in N'$, which is connected and is the set of non-zero idemoptents of S. So by theorem 3.2 again, e^* is the zero for N'. Thus $axe^* = e^* \in N$, contradicts to the definition of local zero.

<u>Corollary</u> : If S is compact and E^* is connected, then $N = \mathrm{Tod}\ e$ for all $e \in E^*$.

This is a consequence of our theorem 3.5 and theorem 1.10.

§4. <u>An example</u>

In this section, we construct an example to show that even if S is locally compact but not locally connected, some important properties concerning the set N, which we have just discussed in section 2 and section 3, are still valid.

<u>Example 4.1.</u> Let S^* be the subset of the plane E^2 consisting of line segments L_n, joining the points $(1,1)$ and $(\frac{1}{n},0)$ for all $n = 1,2,\ldots$. Let $S = S^* - (1,1)$.

The topology of S is the usual topology inherited from the plane.

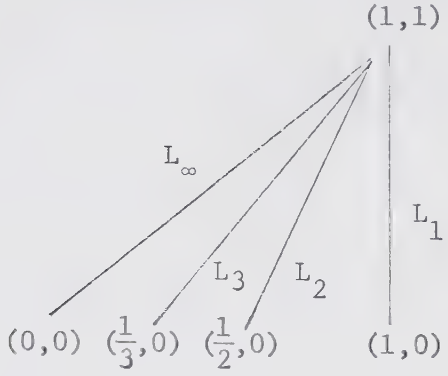


Figure II.

For any point (x_n, y_n) on L_n, we have

$$x_n = \lambda_n \frac{1}{n} + (1-\lambda_n) = \lambda_n(\frac{1}{n} - 1) + 1$$

$$y_n = (1-\lambda_n) \qquad \text{where} \quad 0 \leq \lambda_n < 1 .$$

Define the multiplication $*$ on S as follows:

$$(x_n, y_n) * (x_m, y_m) = (\lambda_n(\frac{1}{n} - 1),\ 1-\lambda_n) * (\lambda_m(\frac{1}{m} - 1) + 1\ ,\ 1-\lambda_m) = (\delta(\frac{1}{mn} - 1) + 1, \sigma\}$$

where $\delta = \min\{\lambda_n, \lambda_m\}$ and $\sigma = \min\{1-\lambda_m, 1-\lambda_n\}$
$= \min\{y_n, y_m\}$.

Clearly, $*$ is associative and S is a semigroup.

To see that S is a mob, we have to verify that $*$ is a continuous mapping from $S \times S$ into S . It suffices for us to check that $*$ is continuous at $(0,0)$, for the continuity at other points is clear.

Suppose $(x_n,y_n) \to (0,0)$, $(x_m,y_m) \to (x,y)$. We have to show that $(x_n,y_n) * (x_m,y_m) \to (0,0) * (x,y) = (0,0)$. Suppose $x_n,y_n) * (x_m,y_m) = (a,b)$. Since $(x_n,y_n) \to (0,0)$. So for all $\epsilon \to 0$, there exists an integer $N_o > \frac{1}{\epsilon}$ such that when $n > N_o$, $x_n < \epsilon$, $y_n < \epsilon$. So $b = \min\{y_n,y_m\} \leq y_n < \epsilon$. Since the point (a,b) lies on the line L_{mn} , we have $a < \frac{1}{n} + \epsilon \leq 2\epsilon$. Hence $(x_n,y_n) * (x_m,y_m) = (0,0)$, which implies that $*$ is continuous at $(0,0)$. So S is a mob. [Note that $*$ is not continuous at (1.1)].

In this example, S is locally compact but not locally connected. The point $(1,0)$ is a local zero for S . $\overline{N} = (0,0) \cup \{(\frac{1}{n},0)|n=2,3,\ldots\}$, which is totally disconnected and has no local zeros of S . $\overline{N}$ is an ideal of S . The component of $(1,0)$ in S is clearly disjoint from $\overline{N}$. The topological zero divisors of $e^* = (1,0)$ is the set $S-L_1$ which is the maximal ideal among the ideals $\{\text{Tod } a | a \in N'\}$. Moreover, Tod e^* is a prime ideal containing N . Since $N' \nsubseteq E^*$, we can see that S is not an N-mob.

Remark : Professor J.M. Day pointed out to me that the mob we constructed in this example is topologically equivalent to the teeth of the comb space with zero added.

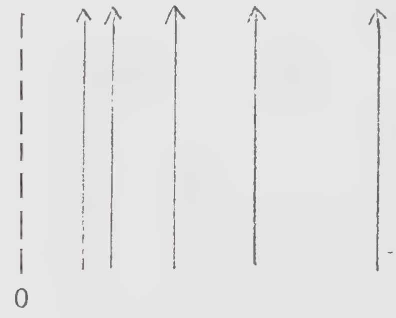


Figure III .
(open at the top).

CHAPTER III

On radicals in non-abelian mobs

In ring theory, in order to get information about the structure theorems, the use of a radical is a basic technique. Many kinds of radicals in rings has been introduced and studied in the literature. We observe that some of these radicals which concern only multiplicative properties can be studied in semigroups. In previous chapters, we have studied some kinds of radicals in abelian mobs. The purpose of the present chapter is to extend our studies on these radicals to non-abelian mobs. We are mainly interested in studying the Wedderburn radical and the Thierrin radical, as these two radicals are closely related to what we have discussed in chapter I and chapter II.

In this chapter, we give a condition for the right annihilator and the left annihilator of the Wedderburn radical W in a compact mob to be contained in W. We show that under a special condition, the celebrated Hopkin's theorem and Levitski's theorem in ring theory can be transferred to compact N-mob without assuming d.c.c. or a.c.c. on it ideals. The notion of e-invariant radical is introduced and we show that in a compact divisible non-abelian mob, some of its e-invariant radical can be a compact connected group.

Finally, the concepts of compressed ideals and Thierrin radicals from ring theory are transferred to mobs. If S is a compact non-abelian mob satisfying Numakura's condition, then the Thierrin radical of our open ideal A in S is w-reducible.

Throughout, a "mob" always means a Hausdorff topological semigroup with zero. We use E to denote the set of all idempotents of S, $|c|$ to denote the cardinal number of the subset c of S, and K to denote the minimal ideal of the compact mob S.

Notations. Let A,B be subsets of a mob S.

$A-B$ = the set theoretic complement of B in A.

$J(A)$ = $A \cup AS \cup SA \cup SAS$, that is, the smallest ideal containing A.

$R(A)$ = $A \cup AS$, that is, the smallest right ideal containing A.

$L(A)$ = $A \cup SA$, that is, the smallest left ideal containing A.

It is obvious that if S and A are compact, then $J(A)$, $R(A)$, $L(A)$ are compact.

$J_o(A)$ = the union of all ideals contained in A, that is, the largest ideal contained in A. (if $J_o(A) \neq \phi$)

$R_o(A)$ = the union of all right ideals contained in A, that is the largest right ideal contained in A. (if $R_o(A) \neq \phi$)

$L_o(A)$ = the union of all left ideals contained in A, that is, the largest left ideal contained in A (if $L_o(A) \neq \phi$).

Koch and Wallace [14], page 681, proved that: If S is compact, then $J_o(A)$, $R_o(A)$ and $L_o(A)$ are open if A is open.

For other terminology and notation, the reader is referred to Chapter I, Chapter II and [21].

§1. Wedderburn radicals.

Let S be a mob with zero. An element x of S is said to be algebraically nilpotent or A-nilpotent if there exists an integer $n \geq 1$ such that $x^n = 0$. A non-zero ideal I of S is said to be A-nil if it consists of only A-nilpotent elements. An ideal I of S is said to be A-nilpotent if there exists an integer $n \geq 1$ such that $I^n = 0$, this means that the set of all products $i_1 i_2 \ldots i_n$ of n elements of I is zero. We use W to denote the set of all A-nilpotent elements of S . The maximal ideal contained in W is called the Wedderburn radical of S ; it is in fact the union of all A-nil ideals of S . In this section, all ideals to be considered are non-zero, thus under this assumption, a compact mob need not contain a minimal ideal

Example 1.1. Let $S = \{a,b,c,d,0\}$ and define the multiplication in S by the following table:

.	a	b	c	d	0
a	a	c	c	a	0
b	d	0	b	0	0
c	a	0	c	0	0
d	d	b	b	d	0
0	0	0	0	0	0

Associativity of the multiplication can be easily verified to confirm that S is a mob. Now $W = \{0,b\}$, which is the set of all A-nilpotent elements of S ,is not an ideal of S .

Remark. In Chapter II , we proved that: The set of all nilpotent elements of a compact mob, in topological sense, is an ideal. This example illustrates that the set of all A-nilpotent elements of a non-abelian compact mob need not be an ideal.

Proposition 1.2. Let S be an abstract semigroup, W be the set of all A-nilpotent elements of S . If $|S-W| = 1$, and W is a subsemigroup of S then W is the Wedderburn radical of S .

Proof : Since $|S-W| = 1$, then we can write $S-W = \{c\} \neq 0$. Suppose if possible, W is not a right ideal of S , then there exists an element $a \in W$ such that $a c \notin W$. Hence $ac = c$, which implies $c = a^n c$ for all integer $n \geq 1$. As $a \in W$, then $a^k = 0$ for some integer $k \geq 1$. This implies that $c = 0$, a contradiction. Our supposition is impossible, W is a right ideal of S . Similarly, we can prove that W is a left ideal of S . W , is therefore, indeed the Wedderburn radical of S .

By the method of N.H. McCoy [16], we can also prove that: Every one-sided A-nil ideal of a mob S is contained in some A-nil ideal of S .

<u>Definition 1.3.</u> Let A be a non-vacuous subset of a mob S . The left annihilator $l(A)$ of A is the set of all $x \in S$ with $xA = 0$. The right annihilator $r(A)$ of A is the set of all $x \in S$ with $Ax = 0$. Clearly, $l(A)$ and $r(A)$ are closed left and right ideals of S respectively. Moreover, $A \subset r(l(A))$ and $A \subset l(r(A))$ and if $\phi \neq A_1 \subset A_2$, then $l(A_1) \supset l(A_2)$ and $r(A_1) \supset r(A_2)$.

<u>Example 1.4.</u> The following shows that the right annihilator (left annihilator) of the set W need not to be contained in W .

Let $S = \{0\} \times [\frac{1}{2},1] \cup [\frac{1}{2},1] \times \{0\}$ with usual topology inherited from the plane. Define a multiplication $*$ in S as follows:

$$(x_1,0) * (x_2,0) = (\max\{\tfrac{1}{2},x_1x_2\}\ ,\ 0)$$
$$(\tfrac{1}{2},y) * (x,0) = (\tfrac{1}{2},0)$$
$$(\tfrac{1}{2},y_1) * (\tfrac{1}{2},y_2) = (\tfrac{1}{2},\ \min\{y_1,y_2\})$$

Where $x \in [\frac{1}{2},1]$, $y \in [0,1]$.

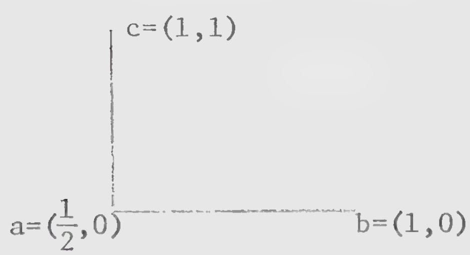


Figure I.

Then S with this multiplication is a mob with zero. Clearly $W = [a,b)$, $r(W) = [a,c)$, $r(W) \not\subset W$.

We are now going to find out the conditions which will give us that $r(W) \subset W$.

Lemma 1.5. Let S be a compact mob such that W is open. If I is any one-sided ideal of S which is not contained in W , then I contains a non-zero idempotent.

Proof : Let I be a right ideal which is not in W . Then we can have $x \in I-W$. Consider the principal right ideal $R(x)$ generated by x . As S is compact and I is a right ideal, then we have $\Gamma(x) = \overline{\{x^n\}_{n=1}^{\infty}} \subset R(x) = x \cup xS \subset I$. By the basic result of compact mobs [21], page 22, there exists an idempotent $e^2 = e \in \Gamma(x) \subset I$. Since $x \notin W$, then $x^k \notin W$ for all integers $k \geq 1$. As W is open, we must have $0 \neq e \in W$. Our proof is completed.

Lemma 1.6. Let S be a compact mob such that the set W is open. Then there is a closed left (right) ideal L minimum with respect to not being contained in W . Moreover, L has the form $L = Sf$ with f a primitive idempotent contained in L .

Proof : Suppose that I is any two-sided ideal which is not contained in W , then by our lemma 1.5, we can find an idempotent $0 \neq e^2 = e \in I-W$. So $SeS \subset SIS \subset I$. SeS is a two-sided ideal of S and is non-A-nil since $e \notin W$. Now, let T be the collection of all closed non-A-nil ideal of S contained in I and let $\{T\alpha\}_{\alpha}$ be the linear ordered subcollection of T .

Then $I_o = \bigcap_\alpha I_\alpha$ is non-empty since S is compact. Further, I_o is an ideal of S contained in I and $I_o \not\subset W$ since W is open and each T_α is compact, $T_\alpha \not\subset W$. Thus $\{T_\alpha\}_\alpha$ has a lower bound and Zorn's lemma assures that the existence of a minimal closed non-A-nil ideal M in I. Applying K. Numakura's argument in [18], page 409, we have $M = SfS$, with f a primitive idempotent of S. Clearly, Sf is a closed left ideal and $Sf \not\subset W$. We claim that Sf is, in fact, the minimal such one. For let L be a non-A-nil closed left ideal in Sf. Then by lemma 1.5, there exists a non-zero idempotent $g^2 = g \in L-W$ so $g \in Sf$ implies $gf = g$, and hence $(fg)^2 = f(gf)\, g = fg$. Thus fg is a non-zero idempotent contained in fSf. Since f is primitive, we have $fg = f$. Thus $fg = f \in fL \subset L$, which implies $L = Sf$. Our proof is completed.

<u>Theorem 1.7.</u> Let S be a compact mob such that the set W is open. If every one-sided non-A-nil ideals of S has a non-zero intersection with W, then $r(W) \cup l(W) \subset W$.

<u>Proof</u> : Suppose on the contrary, $r(W) \not\subset W$. By lemma 1.5, $r(W)$ has an idempotent $e^2 = e \in r(W) - W$. By lemma 1.6, we can find a minimal non-A-nil closed left ideal $L = Sf \subset r(W)$, where f is a primitive idempotent of L. Since $Lf \subset Sf = L$ and Lf is a minimal closed non-A-nil left ideal of S, so by the minimality of L, we have $L = Lf$. But we assume that $L \cap W \neq 0$. Denote $L \cap W = L_1 \neq 0$.

Then we have $L_1 \subset L = Lf$. This implies that for every $a \in L_1$, $a = bf$ with $b \in L$. Hence $af = (bf)f = bf = a$, that is $L_1 f = L_1$. On the other hand, we have $L_1 = L_1 f \subset WL \subset Wr(W) = 0$, a contradiction. Thus we proved that $r(W) \subset W$. Similarly, $l(W) \subset W$. So $r(W) \cup l(W) \subset W$.

It is well-known that in theory of rings, Hopkin and Levitski proved the following celebrated theorem: If a ring R satisfies the d.c.c. (or a.c.c.) on its one-sided ideals, then any nil ideal of R is a nilpotent ideal of R . (Or in other words, under the a.c.c. or d.c.c. on ideals <u>any non-nilpotent ideal of R is non-nil</u>). In a compact mob S , a nil ideal of S need not be nilpotent, this can be seen by the following:

<u>Example 1.8.</u> Let S be the unit interval with the usual multiplication. Then $I = [0,1)$ is an nil ideal of S . (nil in topological sense, see chapter II) , however, I is not a nilpotent ideal, since $I^n = I$ for all n .

<u>Example 1.9.</u> Let $S = [\frac{1}{2},1]$ with the usual topology, the multiplication on S is defined by $x * y = \{\frac{1}{2},xy\}$ where xy is the usual multiplication. Then S is a mob and $[\frac{1}{2},1)$ is an A-nil ideal of S . But $[\frac{1}{2},1)$ is not A-nilpotent.

The following theorem shows that Hopkin's and Levitski's results can be obtained in compact N-mobs under some special conditions.

(For the definition and properties of N-mobs, see chapter II).

Theorem 1.10. Let S be a compact N-mob. If a non-nilpotent ideal I of S contains at least one closed non-nilpotent left (right) ideal of S, then I is non-nil. (Nil and Nilpotent in topological sense).

Proof : Let I be a non-nilpotent ideal of S. Let T be the collection of all closed non-nilpotent left ideals of S contained in I. T is partially ordered by inclusion and is non-empty by our assumption on I. Suppose $\{T_\alpha\}_\alpha$ is a linearly ordered subcollection of T. Then $\cap_\alpha T_\alpha$ is non-empty since S is conpact. So $\cap_\alpha T_\alpha$ is a closed non-empty ideal on I. We claim that $\cap_\alpha T_\alpha$ is non-nilpotent. For if not, then $\cap_\alpha T_\alpha$ is nilpotent and hence is nil. So $\cap_\alpha T_\alpha \subset N$, where N is the set of all nilpotent elements of S. (in topological sense). This implies that the intersection of finitely many members of $\{T_\alpha\}_\alpha$ is contained in N because N is open and T_α's are compact. Since $\{T_\alpha\}_\alpha$ is a $\supset$ chain, we have $T_\alpha \subset N$ for some α. But since T_α is a closed subset of S, then by K.Numakura [20], page 675, T_α is nilpotent. This contradiction establishes our claim. Thus $\{T_\alpha\}_\alpha$ has a lower bound and Zorn's lemma assures the existence of a minimal closed non-nilpotent left ideal, say L_1 in I. We have $L_1^2 \subset L_1$, but since L_1 is non-nilpotent, we have $L_1^2 = L_1$ by the minimality of L_1.

Let $\mathcal{U}$ be the family of all left ideals J in S such that $L_1 J \neq 0$ and $J \subset L_1$. $\mathcal{N}$ is non-empty since $L_1 \in \mathcal{U}$. Since is compact, applying the above arguments and Zorn's lemma, we can prove that $\mathcal{N}$ has a minimal closed left ideal of S , say J_1 such that $L_1 J_1 \neq 0$. Let $0 \neq x \in J_1$ be such that $L_1\ x \neq 0$. $L_1 x$ is a closed left ideal of S . $L_1(L_1 x) = L_1^2 x = L_1 x \neq 0$ and $L_1 x \subset L_1 J \subset L_1$. Hence $L_1 x \in \mathcal{U}$. Moreover, $L_1 x = J_1$ since $L_1 x \subset J_1$ and J_1 is minimal. Now let $a \in L_1$ be such that $ax = x$. then for any integer $n \geq 1$, we have $a^n x = x$, which implies that $a^n \neq 0$. As $a \in L_1 \subset I$, I is therefore non-nil. Our proof is completed.

<u>Corollary 1.</u> The Wedderburn radical of a finite simigroup is A-nilpotent.

<u>Corollary 2.</u> If the Wedderburn radical of a mob S is A-nilpotent, then every non-zero minimal ideal of S is contained in $r(W) \cap l(W)$.

Following R. Baer [1], the radical ideal of an abstract semigroup is defined as follows:

<u>Definition 1.11.</u> A subset Q of an abstract semigroup S is a radical ideal if (1) Q is an ideal of S

(2) Q is an A-nil ideal of S

(3) $\frac{S}{Q}$ has no non-zero A-nilpotent ideals.

If Q is a radical ideal of S , then the structure of the abstract quotient semigroup $\frac{S}{Q}$ has been studied by A.H. Clifford [3]. We observe that the following theorems in ring theory which are due to R.Baer [1] can also be transferred to abstract semigroups.

Theorem 1.12. [Baer] The Wedderburn radical is a radical ideal of an abstract semigroup S .

Proof : See [1]

A semigroup S is said to satisfy Clifford's condition: if every two-sided ideal of S contains at least one non-zero left minimal ideal and at least one non-zero right minimal ideal of S . [3], page 840.

Theorem 1.13. If Q is a radical ideal of an abstract S such that $\frac{S}{Q}$ satisfies Clifford's condition, then Q is the Wedderburn radical of S .

(The proof of this theorem in ring theory can be found in [1] and elsewhere. Because this theorem gives a characterization of radical ideals in mobs, we provide a proof here for the sake of completeness.)

Proof : We only need to show that Q is the union of all A-nil ideals of S .

Suppose on the contrary, J is an A-nil ideal in S such that $J \nsubseteq Q$. Then $Q \cup J$ is an ideal properly containing J. Since $\frac{S}{Q}$ satisfies the Clifford's condition, hence by Clifford [3], page 840, there exists a non-zero minimal ideal M of S such that $Q \overset{\subset}{\neq} M \subset Q \cup J \subset S$. Hence we have $\hat{0} \neq \frac{M}{Q} \subset \frac{Q \cup J}{Q} \subset \frac{S}{Q}$, where $\hat{0}$ is the zero element of the quotient semigroup $\frac{S}{Q}$. As J is A-nil, $\frac{Q \cup J}{Q}$ is A-nil which implies $\frac{M}{Q}$ is A-nil. Thus $\frac{M}{Q}$, being a non-zero minimal ideal of $\frac{S}{Q}$, we have either $(\frac{M}{Q})^2 = \hat{0}$ or $(\frac{M}{Q})^2 = (\frac{M}{Q})$. Since $\frac{M}{Q}$ is A-nil, we cannot have $(\frac{M}{Q})^2 = \frac{M}{Q}$. So we must have $(\frac{M}{Q})^2 = 0$. But this case is excluded, since Q is a radical ideal of S, $\frac{S}{Q}$ has no non-zero A-nilpotent ideals of S. Hence Q contains all A-nil ideals of S and must contain W. On the other hand, as Q is a radical ideal of S, it is contained in W. Thus we conclude that $Q = W$ after all.

It is well known that a compact 0-simple mob S is the union of all 0-minimal left (right) ideals of S [21], page 63. We shall prove that the same result holds in dual mob. Stefan Schwarz [23] called a mob $S \neq 0$ dual if for every left ideal L of S, we have $l(r(L)) = L$ and for every right ideal R of S, we have $r(l(R)) = R$. Examples of discrete dual mobs are given in his paper [23]. He showed that: For every left ideal $L \neq S$ and right ideal $R \neq S$, we have $l(R) \neq 0$ and $r(L) \neq 0$ and if A is an one-sided ideal of S, then $l(A)$ and $r(A)$ are two-sided.

Moreover, let $\{A_v | v \in \Lambda\}$ be a collection of subsets of S ,

then $l(\bigcup_{v\in\Lambda} A_v) = \bigcap_{v\in\Lambda} l(A_v)$, $r(\bigcup_{v\in\Lambda} A_v) = \bigcap_{v\in\Lambda} r(A_v)$ [23].

Theorem 1.12. Let S be a dual mob. If the intersection of all maximal ideals of S is zero, then S is the union of all minimal left (right) ideals of S .

Proof : Let $\Omega = \cup$ all minimal left ideals of S . Ω itself is a left ideal of S . If $\Omega \neq S$, then by Schwarz's result mentioned above, $r(\Omega) \neq 0$. Hence there exists $x = 0$ such that $\Omega x = 0$. Then x belongs to all right annihilaters of all the minimal left ideals of S . That is, x belongs to the intersection of all maximal two-sided ideals of S . By our assumption, we have $x = 0$, which is a contradiction.

§2. The e-invariant radicals.

A.D. Wallace [3] has introduced the concepts of relative ideals in a mob S . He said that a subset $A \subset S$ is a left (right; two-sided) T-ideal of S if $TA \subset A$.$(AT \subset A$, $TAT \subset A)$, where T is a closed submob of S . In this section, we are interested in a particular case, that is when T is only an idempotent e of S . We shall consider the set $ⓔ = \{x \in S | ex = x\}$ and call ⓔ be the e-invariant radical of S . In fact, ⓔ is the union of all minimal left e-ideals of S .

Recall that a mob S is left 0-simple if S does not contain a non-zero proper left ideal of S. A necessary and sufficient condition for a mob S to be left 0-simple is that $Sx = S$ for all non-zero x of S. [21].

<u>Proposition 2.1.</u> Any non-zero element of a compact left 0-simple mob S is in some e-invariant radical of S where $e^2 = e \neq 0$

<u>Proof</u> : Given any non-zero element $a \in S$, we can define $Q = \{x \in S | xa=a\}$. Q is non-empty since S is left 0-simple. S is a compact mob, so clearly, Q is a compact submob of S. Hence, Q contains an idempotent e of S. [21], page 22. We claim that $e \neq 0$ for otherwise $a = 0$. Therefore there exists an idempotent $e \neq 0$ such that $ea = a$. That is, any non-zero element a of S is in ⓔ. for some $0 \neq e = e^2 \in S$.

<u>Proposition 2.2.</u> Let S be a compact Ω-mob containing a maximal proper right ideal R of S. If S is either connected or $|S-R| > 1$, then there is an e-invariant radical of S which is equal to S.

<u>Proof</u> : Let $a \in S-R$. Consider the principal right ideal generated by a. Since $a \notin R$ and S is an Ω-mob, then we have $R \subsetneq a S \cup a$. Since R is a maximal proper right ideal contained in S, so $a S \cup a = S$. Hence either $aS = R$ or $aS = S$. If S is connected, then aS is connected, so $S = aS \cup a$ if and only if $a \in aS$ and hence if $aS = S$.

If $|S-R|>|$, then $aS \neq R$, so $aS = S$. In both cases, we must have $aS = S$. Hence $a^n S = S$ for all $n \geq 1$. Since S is compact, consequently, we have $eS = S$ with $e^2 = e \in \Gamma(a)$. In other words, $(e) = S$.

It is easily seen that if S is a mob with zero, then all e-invariant radicals of S has non-empty intersection. However, we also have the following:

<u>Proposition 2.3.</u> Let S be a compact mob without zero. If E is abelian, then the e-invariant radicals of S have non-empty intersection.

<u>Proof</u> : This result follows easily from the structure of minimal ideal of a compact mob. For let K be the minimal ideal of S , then K is a subgroup of S with identity e , each fSf is a subgroup and intersects with K , so $e \in fSf$ for each idempotent f , hence $e = fe = ef$. Hence $K = eke \subset fekef \subset fSf$ for all $f^2 = f$. This means that $\bigcap_{f \in E} (f) \neq \phi$.

<u>Remark.</u> If E is non-abelian, then the above statement is not true. For example, let $S = [0,\frac{1}{2}] \times [0,\infty)$ with the usual topology inherited from the plane. Define the multiplication $*$ in S by $(x_1,y_1) * (x_2,y_2) = (x_1x_2,x_1y_2+y_1)$. S is easily verified to be a non-abelian mob without zero. E is clearly non-abelian.

The only idempotents of S are $e=(0,0)$, $f = (0,1)$ and $e \cap f = \{e\} \cap \{f\} = \phi$.

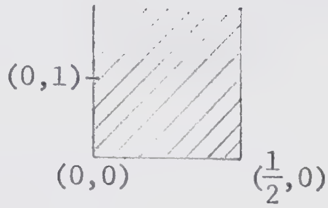


Figure II.

<u>Theorem 2.4.</u> Let S be a compact mob and T be a closed submob of S . If J is a maximal T-ideal of S such that $|S-J| > 1$ and has empty intersection with all e-invariant radicals of S , with $e \in T$, then T must be contained in J .

<u>Proof</u> : Let us denote $S-J=A$. Since J is a maximal T-ideal of S , we have $S=A \cup J$. Hence $TST = T(A \cup J)T = TAT \cup TJT \subset TAT \cup J$. We claim that $TAT \subset J$ for otherwise, $TAT \not\subseteq J$ implies $TAT \cap A \neq \phi$. As TAT is a T-ideal of S , hence $A \subset TAT = S$. Then, for any $a \in A$, we have $a = t_1 a t_2 = t_1^n \, a \, t_2^n$ for all integer $n \geq 1$. By a well known result of Schwarz [24], $a = e\,a\,t'$ where $e^2 = e \in T$, $t' \in T$. Thus $e\,a = e(e\,a\,t') = a$ (*) . By our hypothesis, $A \cap e = \phi$ for all $e \in T$. Consequently, for any $a \in A$, we have $ea \neq a$, which contradicts (*) . Hence we must have $TAT \subset J$ and our claim is established. So $TST \subset J$. By Bednarek and Wallace [2], page 14, we obtain that T J .

If S is a compact connected abelian mob and e is a primitive idempotent of S , then the e-invariant radical os S is the set eSe , which is a compact connected group by the Schutzenberger theorem.

But in general, if S is not abelian, the e-invariant radical of S is not necessarily a compact connected group. For example, let $S = (0,0) \cup \{1\} \times [0,1]$ with usual topology. The multiplication $*$ in S defined by

$$(0,0) * (0,0) = (0,0)$$

$$(1,x) * (1,y) = (1,xy)$$

$$(0,0) * (1,x) = (1,0)$$

$$(1,x) * (0,0) = (0,0) \quad \text{for all} \quad x,y \in [0,1]$$

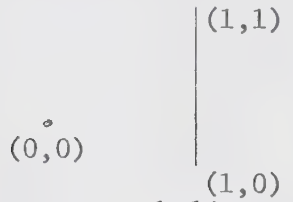


Figure III.

Then S is a compact non-abelian mob. Take $e = (1,1)$, then $e = S$, which is not a compact connected group.

However, in a compact divisible non-abelian mob, it is possible for the e-invariant radical to be a compact connected group. A mob S is said to be divisible if for each $y \in S$, and each integer n, S contains an element x such that $x^n = y$. That is to say, every element in the semigroup has an n^{th} root in the mob for all integer $n \geq 1$.

<u>Theorem 2.5.</u> Let S be a compact divisible mob without zero. Let E be abelian. If e is a primitive idempotent of S, then the corresponding e-invariant radical of S is a compact connected group.

Before proving this theorem, we need the following:

Lemma 2.6. Let S be a compact mob without zero and let E be abelian. If e is a primitive idempotent of S, then the corresponding e-invariant radical of S is a compact group. (Part of this lemma is known from the structure of minimal ideal of a compact mbo, see [21], however, we prove this in detail for the sake of completeness)

Proof : For any $x \in ⓔ$, the set $ⓔx$ is compact. Take $t_1 \in ⓔx$, $t_2 \in ⓔx$, then we have $t_1 t_2 \in ⓔxⓔx = (ⓔxⓔ)x \subset ⓔx$ since $x \in ⓔ$ and ⓔ is a submob of S. This shows that $ⓔx$ is a compact submob of ⓔ. Hence there exists at least one idempotent element of S in the compact submob $ⓔx$, say, $f^2 = f \in ⓔx \subset ⓔ$. We claim that $e = f$. For $f \in ⓔx \subset ⓔ$ implies $ef = f$. As E is abelian, hence $ef = fe = f$. Since we suppose that S is a mob without zero and e is primitive, so we must have $e=f$. Thus for any $x \in ⓔ$, $ⓔx$ contains the idempotent e and hence, there is an element $y \in ⓔ$ such that $yx = e$. That is, y is a left inverse of x in ⓔ. Consequently ⓔ is an abstract group and is a compact space. By the continuity of $ⓔ \times ⓔ \to ⓔ$, $(x,y) \to xy$, implies the mapping of the group ⓔ into itself, $x \to x^{-1}$ is continuous. Hence ⓔ is a compact topological group. In fact, e is the maximal subgroup of S containing the idempotent e.

We are now ready for the proof of our theorem 2.5. First of all, we point out that the group e in a compact divisible mob S is a divisible group. This is the well-known result due to Gelbaum, Kalisch and Olmstead [7]. Recently, J. Mycielski [17] proved that a compact topological group is connected if and only if it is divisible. Applying this result of Mycielski, we obtain that e is a compact connected group.

Theorem 2.7. Any compact connected group can be embedded into a compact divisible 0-simple mob. Any compact divisible 0-simple mob is the union of a zero group and a number of homeomorphic copies of compact connected groups.

Proof : Let G be a compact connected group. By mycielski [17], G is divisible. Let $t \notin G$ and form $S = G \cup \{t\}$. Define a multiplication $*$ in S as follows:

$x * y = xy$, xy is the group multiplication in G, for all x,y in G

$t * t = t$

$t * x = x * t = t$.

Then S is a compact mob with t acting as a zero element of S. Clearly S is divisible and every ideal of S contains t. To prove that S is 0-simple, we let I be an ideal of S different from zero.

Then we have $I \cap G \neq \phi$. Take $x \in I \cap G$, so $G = Gx \subset GI \subset I$. As t is the only element of S not in G and $t \in I$. We have $I = G \cup \{t\} = S$. Hence S is a compact 0-simple mob.

Conversely, let S be a compact 0-simple mob. Then $\{0\}$ is an isolated point of S . [21], page 69. $S-\{0\}$ is a compact simple mob, it is therefore the union of all compact subgroups of S , that is, $S - \{0\} = \cup \{H(e) | e \in E - \{0\} , H(e) = eSe\}$. [21], page 30. It is well known that if S is a compact 0-simple mob, and let e , f be non-zero idempotents of S , then the maximal subgroups $H(e)$ and $H(f)$ containing e and f respectively are iseomorphic compact groups. [21], page 65. Hence S is the union of a zero group and a number of homeomorphic copies of compact groups. Since we assume that S is a compact divisible mob, so by Gelbaum, Kalisch and Olmstead [7], each $H(e)$ is divisible and, by Myclieski [17], each $H(e)$ is a compact connected group. Our proof is completed.

§3. Compressed ideals and Thierrin radicals.

Recall that an ideal P of a mob S is called completely prime, if $ab \in P$ implies that $a \in P$ or $b \in P$. An ideal A of S is called completely semi-prime if $a^2 \in A$ implies $a \in A$. We shall generalize these notions.

Definition 3.1. An ideal A of a mob S is said to be strongly compressed if $a_1^2 a_2^2 \in A$ implies $a_1 a_2 \in A$, where a_1, a_2 are two distinct elements of S .

For instance, let $S = \{0,a,b,c,d,e\}$ a multiplication table as follows:

.	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	d	0	b
b	0	0	0	0	0	0
c	0	e	0	0	b	0
d	0	b	0	0	0	0
e	0	0	0	b	0	0

S can be checked to verify that S with this multiplication is a semigroup. Let $A = \{0,a,d\}$, then A is an ideal of S but not compressed, for $c^2d^2 = 0 \in A$, but $cd = b \notin A$. The ideal $B = \{0,a,b,d,e\}$ is a strongly compressed ideal of S.

Proposition 3.2. Let S be a compact connected mob such that $S^2 \neq S$, then S is the union of strongly compressed ideals of S, each of which is dense in S.

Proof : Since $S \neq S^2$, then we can take an element $a \in S$, which is not in S^2. Clearly, $S-a$ is a maximal proper ideal of S. [9], page 503. Suppose if possible, $a_1^2 a_2^2 \in S-a$ but $a_1a_2 \notin S-a$ where $a_1 \neq a_2$, then $a_1a_2 = a$. But this means that $a \in S^2$. which is false. We therefore obtain that $S-a$ is a strongly compressed ideal of S.

We claim that $|S-S^2| \geq 2$, since S^2 is compact, hence closed and S is connected, hence $S-S^2$ has to have more than one element. So there are at least two distinct elements a,b in $S-S^2$. Hence $S = (S-a) \cup a \subset (S-a) \cup (S-b)$. Thus $S = \bigcup_{a_i \in S-S^2} (S-a_i)$. As S is compact and connected, each of these ideals is dense in S. [14]. Our proof is completed.

Following G. Thierrin [26], we shall define a compressed ideal in a mob S . An ideal A of S is called compressed if and only if $a_1^2 a_2^2 \ldots a_n^2 \in A$ for any n implies $a_1 a_2 \ldots a_n \in A$. In a compressed ideal, we do not require that the a_n's are all distinct. Hence a compressed ideal is strongly compressed, but the converse is not true. The concepts of compressed ideal is, in fact, a generalization of the completely semi-prime ideal. Clearly, every completely prime ideal of a mob S is compressed, and every compressed ideal of S is completely semi-prime. K. Iséki [12] noted that in semi-rings: If an ideal A is completely prime, then $a_1 a_2 \ldots a_n \in A$ implies $a_1^{\ell_1} \ldots a_n^{\ell_n} \in A$ for any positive integers $\ell_1, \ell_2, \ldots \ell_n$ and $a_1^{\ell_1} a_2^{\ell_2} \ldots a_n^{\ell_n} \in A$ implies $a_1 a_2 \ldots a_n \in A$. It is easy to see that the same statement holds for mobs. An example of compressed ideal in a mob S is given below:

<u>Example 3.3.</u> Let $S = [0,1] \times [0,\infty)$ with usual topology inherited from the plane.

The multiplication $*$ in S is defined by $(x_1,y_1) * (x_2,y_2) = (x_1x_2, x_1y_2+y_1)$. Then S is a non-abelian mob and $A = \{0\} \times [0,\infty)$ is a compressed ideal of S .

Theorem 3.4. Let J be a maximal proper ideal of a compact mob S . Then J is a compressed ideal of S if and only if J is a completely prime ideal of S .

Proof : If J is compressed ideal, then J is completely semi-prime ideal. Hence $a \in S-J$ implies $a^2 \in S-J$. By Theorem 2.2.7 of [21], page 66, we have J is a completely prime ideal of S . Conversely, if J is a completely prime ideal of S , then it is clear that J is compressed.

Definition 3.5. Let A be an ideal of S . The compressed ideal generated by A is called the Thierrin radical of A . In other words, the Thierrin radical of A is the minimal compressed ideal of S containing the ideal A .

Given an ideal A of S . The Thierrin radical of A can be constructed in the following manner: We call an element x a t-element for A if $x = x_1x_2 \ldots x_n$ such that $x_1^2\, x_2^2 \ldots x_n^2 \in A$ for some $n \geq 1$. Let us denote by $T^1(A)$ the set of all t-elements for A , and let $T_1(A) = J(T^1(A))$, that is, the principal ideal generated by the set $T^1(A)$.

Write $T_2(A) = J(T^1(T_1(A))$. By induction, we have $T_n(A) = J(T^1(T_{n-1}(A))$. Then each $T_n(A)$ is an ideal of S and $T_n(A) \subset T_{n+1}(A)$. The Thierrin radical $T^*(A)$ of A is the union of all sets $T_n(A)$ $(n=1,2,\ldots)$, that is, $T^*(A) = \bigcup_{n=1}^{\infty} T_n(A)$. It is clear that an ideal A of S is compressed if and only if itself is the Thierrin radical of A .

Proposition 3.6 (Iséki). The Thierrin radical $T^*(A)$ of an ideal A of a mob S is the intersection of all compressed ideals of S containing A .

(Iséki proved this statement in semi-rings [12], it is trivial to see that the same results holds for mobs)

Proposition 3.7. The algebraic radical of an ideal A in a mob S is contained in the Thierrin radical of A . (For the definition of algebraic radical, see chapter I)

Proof : Let a be a element in the algebraic radical of A . Then there exists an integer $n \geq 1$ such that $a^k \in A \subset T^*(A)$. Since $T^*(A)$ is compressed, we have $a \in T^*(A)$. Hence the algebraic radical of A is contained in $T^*(A)$. If S is abelian, the algebraic radical and the Thierrin radical coincides.

The following condition is called Numakura's condition in a mob S:

For $e, f \quad E$, $J_o(S-e) = J_o(S-f)$ implies that $R_o(S-e) = R_o(S-f)$ and $L_o(S-e) = L_o(S-f)$. [20], page 678.

Theorem 3.8. Let S be a non-abelian compact mob. If S satisfies Numakura's condition, then the Thierrin radical of an open ideal A of S is w-reducible, that is, $T^*(A)$ is the intersection of all completely open prime ideals of S containing A .

Proof : Denote the open completely prime ideals of S containing A by P_α's . Since every completely prime ideal of S is compressed and $T^*(A)$, the Thierrin radical of A , is the smallest compressed ideal containing A , then we have $T^*(A) \subset \underset{\alpha}{\cap} P_\alpha$. Suppose if possible, $T^*(A) \subsetneqq \underset{\alpha}{\cap} P_\alpha$, then by the argument of lemma 3.3 in our chapter I, we can prove that there exists an idempotent $e^2 = e \in \underset{\alpha}{\cap} P_\alpha$, but $e \notin A$. As A is an ideal, we have $J_o(S-e) \supset A$. By Numakura [20], $J_o(S-e)$ is an open prime ideal of S . As we assume that S satisfies the Numakura's condition, hence by Theorem 5 of [20], page 679, the prime ideal $J_o(S-e)$ is in fact completely prime. So we conclude that $\underset{\alpha}{\cap} P_\alpha \subset J_o(S-e)$, which implies $e \notin \underset{\alpha}{\cap} P_\alpha$. This contradiction prove that $T^*(A) = \underset{\alpha}{\cap} P_\alpha$. So $T^*(A)$ is, in fact, w-reducible.

Corollary 1. Under the given assumption, our Thierrin radical is in fact the algebraic radical of A. [see chapter I].

Corollary 2. Let S be a compact mob satisfying Numakura's condition and let C be a compressed ideal of S containing an open ideal A , then C is not the Thierrin radical of A if and only if C contains an idempotent not in A .

Proof : If C is not $T^*(A)$, then applying the result of lemma 3.3 of chapter I, we can find an idempotent $e^2 = e \in C-A$. Conversely, suppose that there is an idempotent $e^2 = e \in C-A$, then by our theorem, we have $T^*(A) \subset J_o(S-e)$. Since $e \notin J_o(S-e)$, we conclude that $e \notin T^*(A)$. Consequently $C \neq T^*(A)$.

Definition 3.9. A subset M of a mob S is said to be M-system if for $a,b \in M$, there exists $x \in S$ such that $axb \in M$. By Numakura [20], an ideal P of S is prime if and only if $S-P$ is an M-system of S .

In a non-abelian mob, a prime ideal of S is not necessarily a completely semi-prime ideal. For example, let $S = \{e_1,e_2,a,b,0\}$ with multiplication table

$\cdot$	e_1	e_2	a	b	0
e_1	e_1	0	0	b	0
e_2	0	e_2	a	0	0
a	a	0	0	e_2	0
b	0	b	e_1	0	0
0	0	0	0	0	0

Then $\{0\}$ is a prime ideal of S . But $b^2 = 0$, $b \not\in \{0\}$. However, the following theorem, which is essentially due to Iséki [12], proved that, under some very special condition, a prime ideal of S can be completely semi-prime.

Theorem 3.10. Let A be a compressed ideal of a mob S . If P is a minimal prime ideal of S containing A , then P is a completely semi-prime ideal of S .

Proof : Suppose that $S-P$ is not empty. Then $S-P$ is an M-system which is the maximal M-system which does not meet A . Let $C(P)$ be the set of all elements of $a = x_1^{m_1}, x_2^{m_2} \ldots x_k^{m_k}$, where $x_1x_2 \ldots x_k \in S-P$, $m_1, m_2, \ldots, m_k$ are all positive integers and $k = 1,2,\ldots$. Clearly, $S-P \subset C(P)$. To prove that $C(P)$ is an M-system, let $b = y_1^{n_1} y_2^{n_2} \ldots y_\ell^{n_\ell} \in C(P)$ and $y_1y_2 \ldots y_\ell \in S-P$. As $S-P$ is an M_o-system, there is an element t of S such that $(x_1x_2 \ldots x_k)\, t\, (y_1y_2 \ldots y_\ell) \in S-P$. Therefore, by the definition of $C(P)$, we can have $x_1^{m_1} x_2^{m_2} \ldots x_k^{m_k}\ t\ y_1^{n_1} y_2^{n_2} \ldots y_\ell^{n_\ell} \in C(P)$. This shows that $axb \in C(P)$ and $C(P)$ is an M-system. We shall prove that $C(P) \cap A = \phi$. Suppose on the contrary, $C(P) \cap A \neq \phi$, then there is an element $c \in A \cap C(P)$. Hence $C = z_1^{t_1} z_2^{t_2} \ldots z_n^{t_n}$ with $z_1z_2 \ldots z_n \in S-P$. Since A is compressed, we have $z_1z_2 \ldots z_n \in A$. That is, $A \cap (S-P) \neq \phi$ which is a contradiction. Thus we must have $C(P) \cap A = \phi$.

This implies that $C(P) \subset S-P$. Hence $C(P) = S-P$. Now let $x^2 \in P$. Suppose that $x \notin P$, then $x \in S-P$. But by the definition of $C(P)$, we have $x^2 \in C(P) = S-P$, which is false. Therefore $x \in P$, P is a completely semi-prime ideal of S.

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